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Dynamic analysis and chaos synchronization of the fractional-order complex Lorenz system

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Abstract : Nowadays, the fractional differential calculus has been applied to the study of dynamic systems. Chaos has been observed in many fractional-order systems, so when a fractional-order system is chaotic and how to synchronize the fractional-order chaotic systems have been two very important problems. In the present manuscript, the author studied the complex Lorenz system which a fractional-order system may exhibit a chaotic behavior could only be analyzed by simulation results. This paper applies the stability theory of fractional-order systems in their dynamic analysis and obtains some useful conclusions. When it comes to the synchronization problems, active control method has been studied.
Keywords: Chaos synchronization, Complex chaotic system, Fractional derivative, Fractional-order complex Lorenz system, Active control method.

1. Introduction

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with correspondence between Leibniz and L'Hospital in 1695, where the meaning of derivative of order one half was discussed [1]. Although fractional calculus has a 300-year-old history, its applications to physics and engineering are just a recent focus of interest. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [2], dielectric polarization [3], electrode-electrolyte polarization [4], electromagnetic waves [5], quantitative finance [6] and quantum evolution of complex systems [7].Chaotic regions are found in a wide scope of parameter space. Different motions along the routes to chaos are analyzed and depicted by bifurcation diagram, LLE and phase portrait. From the point of view of stability theory of the fractional-order system, a new way is discussed to search the lowest order in fractional-order system.

Since the synchronization of chaotic dynamical systems was observed by Pecora and Carroll [8] in 1990, theoretical as well as experimental research on chaos synchronization has been carried out in a variety of nonlinear dynamic systems. It is because chaos synchronization can be applied in vast areas of physics, engineering science, and in particular in secure communication. A variety of approaches have been proposed for the synchronization of chaotic systems, such as complete synchronization [9], anti-synchronization[10], generalized synchronization [11] and projective synchronization [12].

However, until now, all of the studies about fractional-order systems had been based on the state variables in real space, and complex systems are not involved. It is known that complex systems can be widely applied to describe a variety of physical phenomena, such as detuned laser systems, amplitudes of electromagnetic fields, and thermal convection of liquid flows [13–15], etc. In recently years, many research results have been proposed about the dynamic properties in real space and complex space [16–23]. So it is an interesting and meaningful topic for researchers to explore the dynamic behaviors in fractional-order complex nonlinear systems.

Motivated by the above discussion, in this article, the fractional-order complex Lorenz system as a novel dynamic system is firstly proposed. Dynamic behaviors are numerically investigated with varying the system parameters and the fractional derivative orders. Based on the above results, furthermore, chaos synchronization in fractional-order complex Lorenz systems is studied.

2. Fractional derivative and its definitions

Fractional calculus is a generalization of integration and differentiation to a non-integer order integro-differential operator ${}_aD_t^\alpha$ which is defined by

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & R(\alpha) > 0, \\ 1 & R(\alpha) > 0, \\ \int_a^t (d\tau)^{-\alpha} & R(\alpha) > 0, \end{cases} \tag{1} $$

where α is the derivative order, which can be a complex number, $R(\alpha)$ is the real part of α. The numbers a and t are the limits of the operator. There are many definitions for the general fractional derivative. The three most frequently used ones are: the Grunwald-Letnikov definition, the Riemann-Liouville and the Caputo definitions.

The Grunwald–Letnikov (GL)definition derivative with fractional-order α is described by

$$ {}_a^{GL}D_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{2} $$

where the symbol $[\bullet]$ means the integer part.

The Riemann–Liouville (RL) definition of fractional derivatives is given by

$$ {}_a^{RL}D_t^\alpha f(t) = \frac{d^n}{dt^n} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{3} $$

where $n-1 < \alpha < n$ and $\Gamma(\bullet)$ is the gamma function.

The Caputo (C) fractional derivative is defined as follows:

$$ {}_a^{C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{4} $$

where $n-1 < \alpha < n$ and $\Gamma(\bullet)$ is the gamma function.

It is well known that the initial conditions for the fractional differential equations with Caputo derivatives take on the same form as those for the integer-orderones, which is very suitable for practical problems. Therefore, author use the Caputo definition for the fractional derivatives in this paper and notation D^α for ${}_a^CD_t^\alpha$.

3. The fractional-order complex Lorenz system (FOCLS)

Flower et al. introduced the complex Lorenz system [24], and its dynamical property was studied by Mahmoud et al. [25]. This system is useful to describe and simulate the physics of liquid flows. The integer-order complex Lorenz system is expressed as

$$\frac{dx_1'}{dt} = a_1(x_2' - x_1'),$$

$$\frac{dx_2'}{dt} = a_2x_1' - x_2' - x_1'x_3', \quad \frac{dx_3'}{dt} = \frac{1}{2}(\bar{x}_1'x_2' + x_1'\bar{x}_2') - a_3x_3', \quad (5)$$

where $x' = [x_1', x_2', x_3']^T$ is the state vector variable, $x_1' = x_1 + ix_2$ and $x_2' = x_3 + ix_4$ are complex variables while $x_3' = x_5$ is real variable and $i = \sqrt{-1}$. In system (5), a_1 is the Prandtl number, a_2 is the Rayleigh number and a_3 is the size of the region approximated by the system.

The FOCLS is given by

$$D^\alpha x_1' = a_1(x_2' - x_1'),$$

$$D^\alpha x_2' = a_2x_1' - x_2' - x_1'x_3',$$

$$D^\alpha x_3' = \frac{1}{2}(\bar{x}_1'x_2' + x_1'\bar{x}_2') - a_3x_3'. \quad (6)$$

4. Dynamic analysis of the FOCLS

In this section, the dynamic analysis of the FOCLS will be investigated, including real version, symmetry and invariance, equilibrium points, stability and chaotic attractors of system (6).

4.1 Real version

Since the Caputo fractional derivative operator (4) is linear, the system (6) can be written in real version as

$$D^\alpha x_1 = a_1(x_3 - x_1),$$

$$D^\alpha x_2 = a_1(x_4 - x_2),$$

$$D^\alpha x_3 = a_2x_1 - x_3 - x_1x_5,$$

$$D^\alpha x_4 = a_2x_2 - x_4 - x_2x_5,$$

$$D^\alpha x_5 = x_1x_3 + x_2x_4 - a_3x_5. \quad (7)$$

4.2 Symmetry and invariance

Symmetry about x_5-axis, due to the invariance of equations under the change $(x_1, x_2, x_3, x_4, x_5) \to (-x_1, -x_2, -x_3, -x_4, x_5)$. Hence, if $(x_1, x_2, x_3, x_4, x_5)$ is a solution of chaotic system (7), then $(-x_1, -x_2, -x_3, -x_4, x_5)$ is also a solution of the same system (7).

4.3 Equilibrium points

The equilibrium points of system (7) can be calculated by solving the equations $D^\alpha x_j = 0,\ j = 1, 2, 3, 4, 5$ and system (3) have an isolated equilibrium point $E_0 = (0, 0, 0, 0, 0)$ and nontrivial equilibrium points $E_\theta = (r\cos\theta,\ r\sin\theta,\ r\cos\theta,\ r\sin\theta,\ x_5)$, where $r = \sqrt{a_3x_5}$, $x_5 = a_2 - 1$, $\theta \in [0, 2\pi]$. It is obvious that the nontrivial equilibrium points exist when $a_2 > 1$.

4.4 Stability

As to the equilibrium E_0, it is stable when $a_2 < 1$ and unstable when $a_2 > 1$. For E_θ, the characteristic polynomial of Jacobian matrix when $a_2 > 1$ is

$$z \cdot (z + a_1 + 1) \cdot \left(z^3 + (1 + a_1 + a_3)z^2 + (a_1a_3 + a_2a_3)z + 2a_1a_3(a_2 - 1)\right) = 0. \quad (8)$$

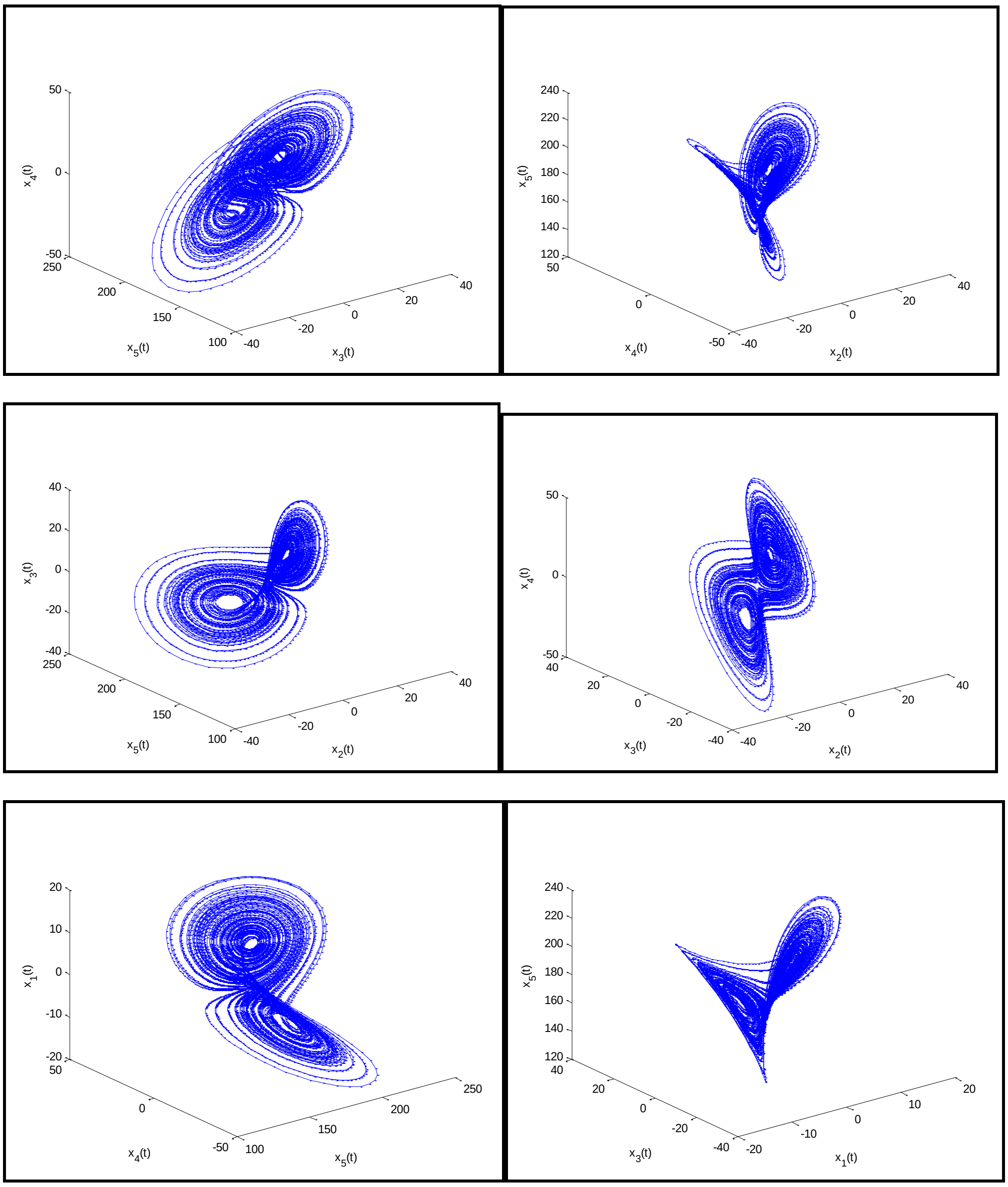
According to the fractional-order Routh–Hurwitz conditions [26], when

$$(1+a_1+a_3)\cdot(a_1a_3+a_2a_3) > 2a_1a_3(a_2-1), \quad (9)$$

E_θ will be stable. But one should note that the condition of inequality (9) is sufficient but not necessary. According to the stability theory of fractional-order systems, the stability region of fractional system is bounded by a cone with vertex at the origin, and extends into the right half of the s-plane such that it enclose an obtuse angle $(\alpha\pi/2,\ 2\pi-\alpha\pi/2)$, where fractional derivative order $0<\alpha<1$. So when one of eigenvalues of equation (8) is in the right half-plane but inside the stability region, the fractional-order system is stable, whereas the corresponding integer-order system is unstable.

4.5 Chaotic attractors

When we take the parameters value $a_1=10,\ a_2=180,\ a_3=1$ and initial condition $x'(0)=[2+3i,\ 5+6i,\ 9]^T$, the system (7) possesses the chaotic attractors which are described by Fig. 1 at fractional-order $\alpha=0.95$.



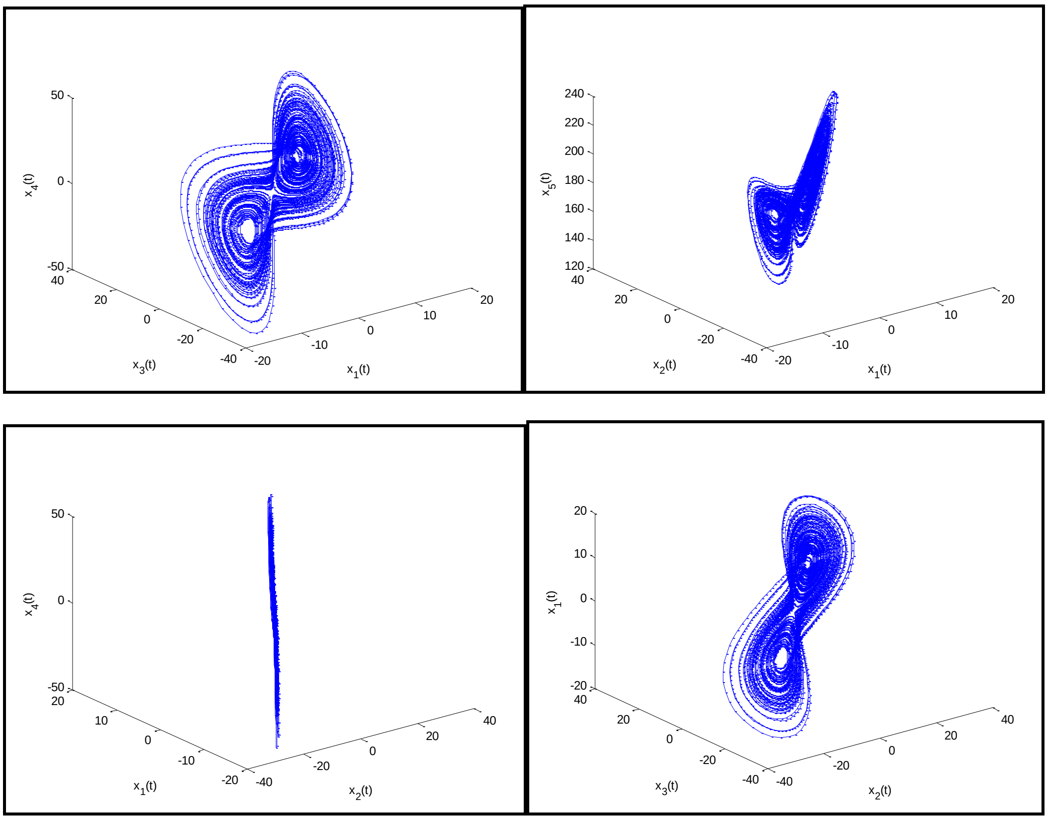


Fig. 1 Phase portraits of the FOCLS.

5. Chaos synchronization of the FOCLS

In this section, chaos synchronization of the FOCLS discussed. Let us define the system (7) as a drive system. Then response system with control function $u(t)=[u_1(t),\, u_2(t),\, u_3(t),\, u_4(t),\, u_5(t)]^T$ is given by

$$
\begin{aligned}
D^{\alpha} y_1 &= b_1(y_3 - y_1) + u_1, \\
D^{\alpha} y_2 &= b_1(y_4 - y_2) + +u_2, \\
D^{\alpha} y_3 &= b_2 y_1 - y_3 - y_1 y_5 + +u_3, \\
D^{\alpha} y_4 &= b_2 y_2 - y_4 - y_2 y_5 + u_4, \\
D^{\alpha} y_5 &= y_1 y_3 + y_2 y_4 - b_3 y_5 + u_5. \qquad (10)
\end{aligned}
$$

In order to estimate the control functions $u_1(t)$, $u_2(t)$, $u_3(t)$, $u_4(t)$ and $u_5(t)$, let us define the error functions between the system (10) and the system (7) as

$$e_1 = y_1 - x_1,\ e_2 = y_2 - x_{2,},\ e_3 = y_3 - x_{3,},\ e_4 = y_4 - x_{4,},\ e_5 = y_5 - x_5 \qquad (11)$$

One can get the error system as follows

$$
\begin{aligned}
D^{\alpha} e_1 &= b_1(e_3 - e_1) + (b_1 - a_1)(x_3 - x_1) + u_1, \\
D^{\alpha} e_2 &= b_1(e_4 - e_2) + (b_1 - a_1)(x_4 - x_2) + u_2, \\
D^{\alpha} e_3 &= b_2 e_1 - e_3 + (b_2 - a_2)x_1 - e_1 e_5 - e_1 x_5 - e_5 x_1 + u_3, \\
D^{\alpha} e_4 &= b_2 e_2 - e_4 + (b_2 - a_2)x_2 - e_2 e_5 - e_2 x_5 - e_5 x_2 + u_4, \\
D^{\alpha} e_5 &= -b_3 e_5 + (a_3 - b_3)x_5 + e_1 e_3 + e_2 e_4 + e_1 x_3 + e_3 x_1 + e_2 x_4 + e_4 x_2 + u_5. \qquad (12)
\end{aligned}
$$

Defining the active control functions $u_j(t)$ as

$$u_1 = V_1 - (b_1 - a_1)(x_3 - x_1),$$
$$u_2 = V_2 - (b_1 - a_1)(x_4 - x_2),$$
$$u_3 = V_3 - (b_2 - a_2)x_1 + e_1e_5 + e_1x_5 + e_5x_1,$$
$$u_4 = V_4 - (b_2 - a_2)x_2 + e_2e_5 + e_2x_5 + e_5x_2,$$
$$u_5 = V_5 - (a_3 - b_3)x_5 - e_1e_3 - e_2e_4 - e_1x_3 - e_3x_1 - e_2x_4 - e_4x_2, \quad (13)$$

where the terms $V_j(t)$ are linear functions of the error terms $e_j(t)$. Now, the error system (12) is reduced to

$$D^\alpha e_1 = b_1(e_3 - e_1) + V_1,$$
$$D^\alpha e_2 = b_1(e_4 - e_2) + V_2,$$
$$D^\alpha e_3 = b_2e_1 - e_3 + V_3,$$
$$D^\alpha e_4 = b_2e_2 - e_4 + V_4,$$
$$D^\alpha e_5 = -b_3e_5 + V_5. (14)$$

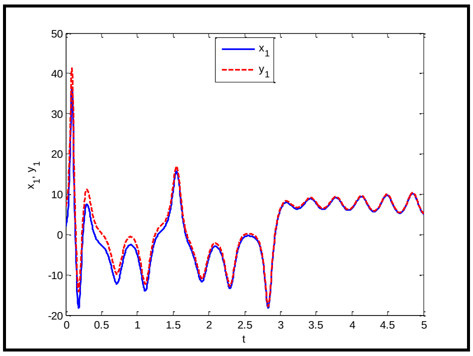
Let us design an appropriate feedback control which stabilizes the system so that $e_j(t)$, $j = 1, 2, 3, 4, 5$ converge to zero as time t becomes large. There are many possible choices for the control inputs $V_j(t)$. We choose $V(t) = Ae(t)$, where $V(t) = [V_1(t), V_2(t), V_3(t), V_4(t), V_5(t)]^T$, A is 5×5 constant matrix and $e(t) = [e_1(t), e_2(t), e_3(t), e_4(t), e_5(t)]^T$. In order to make the closed loop system stable, the matrix should be selected in such a way that the feedback system has eigenvalues λ_j of A satisfy the condition $|\arg(\lambda_j)| > \alpha\pi/2$, $j = 1, 2, 3, 4, 5$. There is no unique choice for matrix A. Let the matrix A is chosen in the form

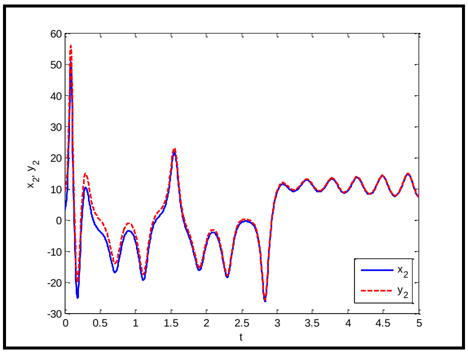
$$A = \begin{bmatrix} -1+b_1 & 0 & -b_1 & 0 & 0 \\ 0 & -1+b_1 & 0 & -b_1 & 0 \\ -b_2 & 0 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1+b_3 \end{bmatrix}.$$

In this particular choice, the error system (14) becomes

$$D^\alpha e_j = -e_j \ , \ j = 1, 2, 3, 4, 5 \ (15)$$

The closed loop system (15) has the eigenvalues -1, -1, -1, -1 and -1. This choice will lead to the error states $e_j(t)$, $j = 1, 2, 3, 4, 5$ converge to zero as time tends to infinity and thus the synchronization of the FOCLS achieved.





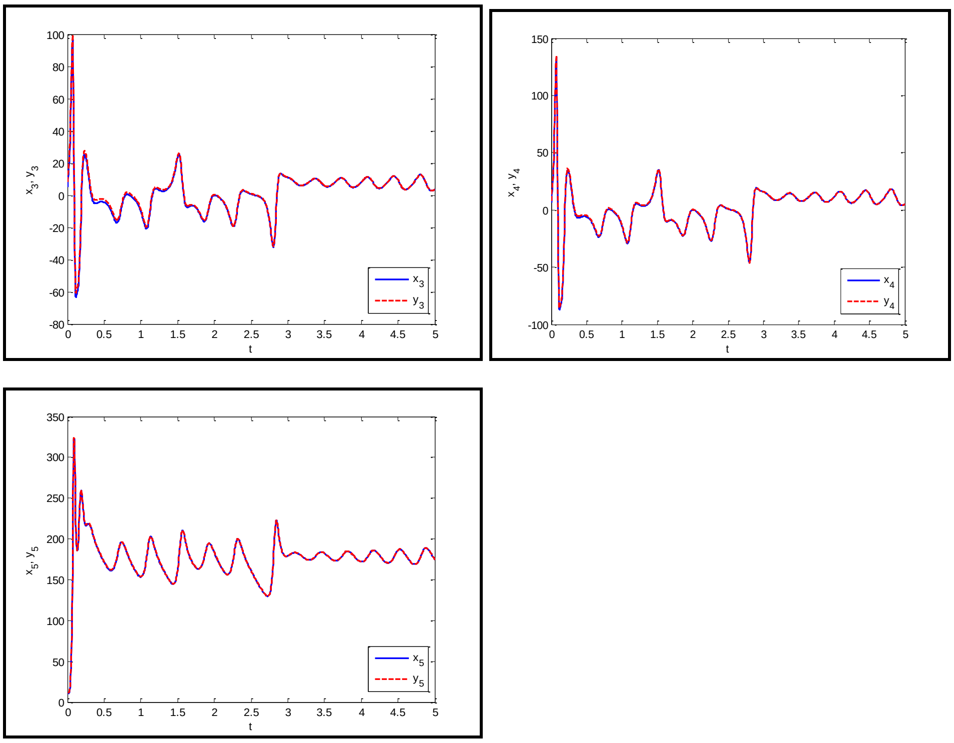


Fig. 2 State trajectories of the drive and response systems.

5.1 Simulation Results

In this section, to verify and demonstrate the feasibility of chaos synchronization of the FOCLS, author obtains the simulation results of the considered systems in complex space at the fractional-order $\alpha = 0.95$. During synchronization, the values of the parameters remain unchanged. The initial conditions are taken as $[2+3i,\ \ 5+6i,\ \ 9]^T$ and $[7+9i,\ \ 8+8i,\ \ 10]^T$ hence the initial error is $[5+6i,\ \ 3+2i,\ \ 1]^T$. Figure 2 display the time response of the states $x_j(t)$ and $y_j(t)$ of the drive system (7) and the response system (10). Figure 3 shows that the error vectors asymptotically converge to zero as time becomes large which implies that chaos synchronization of the FOCLS.

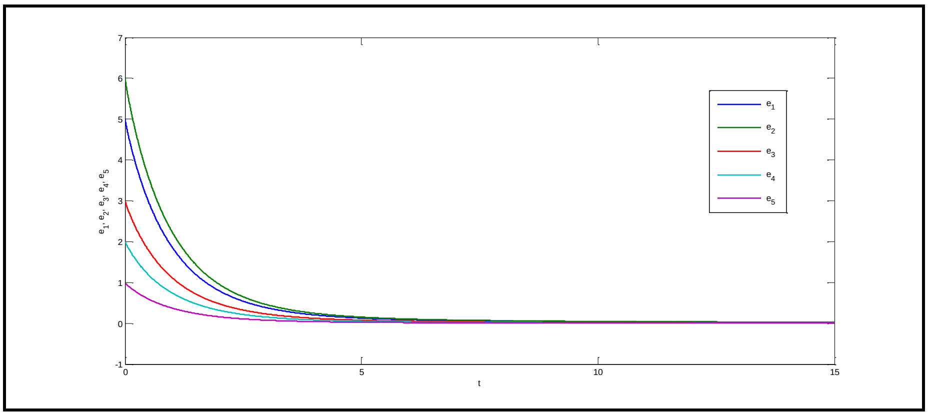


Fig. 3 Error plot of drive and response systems.

6. Concluding remarks

The authors have successfully used the active control method to achieve perfect control of a pair of FOCLS along a desired trajectory, which clearly exhibits the reliability and potential of the method even for fractional order complex systems to be synchronized. The most important part of the study is the dynamical behavior of the FOCLS is investigated.

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