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A new Computational Algorithm to Nonlinear Model of Heat Conduction in the Human Head

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Abstract : A mathematical modelling of multi-layered human skin and subcutaneous tissues (SST) is discussed. The proposed model is based on singular nonlinear boundary value problem (BVP) and also predicts the solution of heat conduction for the temperature distribution in human forehead. To the best of our knowledge until there is no rigorous Legendre computational method has been addressed for the model. The obtained numerical results are compared with finite difference method (FDM). The numerical results were investigated similar to clinical and computational studies.
Keywords: Mathematical modelling; boundary value problem; Legendre computational method.

1. Introduction

The skin in the human body plays a vital role by regulating the exchange of thermal energy with the environment. Several layers of the skin are responsible for maintaining the necessary functional and tissue responses. The human thermoregulatory system gets primarily disturbed due to unstable surrounding environment temperature. This can cause hyperthermia or hypothermia in the body core, and tissue necrosis to the body peripherals. Richardson and Whitelaw [1] predicted the temperature profiles in the biological tissues by keeping skin surface as functions of temperature. The temperature distribution has been analysed by Flesch [2] assuming a heat generation rate has been estimated as an explicit function of the radial distance and an implicit function of the environment temperature in the heat equation (1). By using variational finite element method with respect to various environmental temperatures, Khanday and Saxena [3-5] calculated the mass and temperature distribution at multi-layered skin and sub-dermal tissues. Also, the thermostat phenomenon of brain tissue and estimated the cold stress at multi-layered human head with respect to ambient temperatures was studied.

In recent years, theoretical analysis of the distribution of temperature on the skin surface for various parts of the body has been attracted by the scientists and engineers. It is represented as a boundary value problem. The differential equation of heat conduction is given in Eq. (1). The boundary conditions are represented in Eq. (2).

$$k\frac{d^2T}{dr_d^2}+\frac{2k}{r_d}\frac{dT}{dr_d}+Q=0 \qquad (1)$$

$$lim_{r_d\to 0^+}\frac{dT}{dr_d}=0$$

and

$$k\left(\frac{dT}{dr_d}\right)_{r_d=R}=E\left(T_p-T_s\right) \qquad (2)$$

In recent years, mathematical modelling for the estimation of temperature distribution in human body has been a widely studied research area. This paper proposes to introduce the Legendre Computation Method (LCM) [6] for obtaining the numerical solution of the singular boundary value problem that determines the heat conduction at dermal layers. The current work attempts to study the distribution of temperature at deep dermal layers for heterogeneous thermal conductivity as a function of temperature. The focus of this paper is to compare the results obtained using the finite difference method and Legendre computational method.

Hariharan [7] introduced an efficient Legendre wavelet based approximation method for a few-Newell and Allen-Cahn equations. Hariharan and Kannan [8] reviewed the wavelet solutions for the solutions of reaction-diffusion equations (RDEs) arising in science and engineering. Excellent references are in [9-11].

The organisation of the paper is as follows: Firstly, the mathematical formulation of the model is discussed. Then, a few properties of Legendre computational method are briefly explained. Thereafter, the equation is solved using the LCM and the simulation results are plotted considering four limiting cases. Finally, concluding remarks are presented.

2. Mathematical formulation of the model:

The temperature distribution in various parts of the human body has been a widely researched domain in the past. Initially, Pennes [12] considered the heat transfer in biological systems. The following equation represents the mathematical model of heat transfer in human dermal regions. It is the differential equation of heat conduction.

$$\rho c\frac{\partial T}{\partial t}=k(T)\,\nabla^2 T+k'(T)\nabla .T+Q \qquad (3)$$

The above equation is modified to the one given below in case of steady state processes,

$$k_0\frac{d^2T}{dr_d^2}+\frac{2k_0}{r_d}\frac{dT}{dr_d}+\frac{nk_0}{T-T_p}\left(\frac{dT}{dr_d}\right)^2+\frac{Q}{(T-T_p)^n}=0 \qquad (4)$$

Now, the value of Q in the equation (4) is given as

$$Q = q(37-T) \qquad (5)$$

Replacing (5) in (4), the equation is now written as

$$k_0\frac{d^2T}{dr^2}+\frac{2k_0}{r}\frac{dT}{dr}+\frac{nk_0}{T-T_p}\left(\frac{dT}{dr}\right)^2+\frac{q\left(37-T\right)}{(T-T_p)^n}=0 \qquad (6)$$

with the boundary conditions given by,

$$\left.\frac{dT}{dr_d}\right|_{r_d=0} = 0 \quad , \quad T(R) = T_p \tag{7}$$

As the skin surface is uneven, the equation (6) is converted into a non- dimensional equation using the following transformations,

$Y = T - T_p$ and $t = \frac{r_d}{R}$

where, $0 < t < 1$.

The boundary value problem thus obtained as in equation (8) is solved using LCM, with the value of *n* is considered as zero. For solving the equation with FDM, it was considered to be 0.33 [10]. The solution of singular nonlinear boundary value problems with FDM have been detailed in [13-15].

$$k_0\frac{d^2Y}{dt^2} + \frac{2k_0}{t}\frac{dY}{dt} + \frac{nk_0}{Y}\left(\frac{dY}{dt}\right)^2 + \frac{q\left(37 - T - T_p\right)R^2}{Yn} = 0 \tag{8}$$

with the boundary conditions as:

$$\left.\frac{dY}{dt}\right|_{t=0} = 0 \quad , \; Y(1) = 0$$

Simplifying equation (8), by substituting n=0,

$$k_0\frac{d^2Y}{dt^2} + \frac{2k_0}{t}\frac{dY}{dt} + q\left(37 - T - T_p\right)R^2 = 0 \tag{9}$$

3. Some properties of the shifted Legendre polynomials

The properties of the well known Legendre polynomials $P_n(z)$, defined on the interval [-1,1], are the following [16]:

$$P_n(z) = (-1)^n P_n(z), \quad P_n(-1) = (-1)^n, \quad P_n(1) = 1 \tag{10}$$

It is of common knowledge that the weight function is $\omega(z) = 1$ and the weighted space $L^2_\omega(-1,1)$ is equipped with the following inner product and norm;

$$(u,v) = \int_{-1}^{1} u(z)v(z)\omega(z)dz, \quad \|u\| = (u,u)^{\frac{1}{2}}. \tag{11}$$

The set of Legendre polynomials forms a complete orthogonal system $L^2(-1,1)$ and;

$$\|P_n(z)\|^2 = h_n = \frac{2}{2n+1}, \tag{12}$$

is obtained. In order to use these polynomials on the interval [0, L] the so-called shifted Legendre polynomials are defined by introducing the change of variable $z = \frac{2x}{L} - 1$.

The shifted Legendre polynomials are defined as;

$$P_n^*(x) = P_n\left(\frac{2x}{L} - 1\right) \text{ where } P_n^*(0) = (-1)^n, \tag{13}$$

The analytic form of the shifted Legendre polynomial $P_n^*(x)$ of degree n is given by;

$$P_n^*(x) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!}{(n-k)!(k!)^2 L^k} x^k \ . \tag{14}$$

Assume $\omega_L(x) = 1,$ and the weighted space $L^2_{\omega_L}(0, L)$ be defined with the following inner product and norm;

$$(u, v)_{\omega_L} = \int_0^L u(x)\, v(x)\, \omega_L(x)\, dx, \quad \|u\|_{\omega_L} = (u, u)_{\omega_L}^{\frac{1}{2}} . \tag{15}$$

The set of the shifted Legendre polynomials forms a complete $L^2_{\omega_L}(0, L)$ orthogonal system and $\left\|P_n^*(x)\right\|^2_{\omega_L} = \frac{L}{2} h_n = \frac{L}{2n+1}$ is obtained. The function u(x) which is square integrable in [0,L], may be written in terms of shifted Legendre polynomials as;

$$u(x) = \sum_{i=0}^{\infty} c_i\ P_i^*(x), \tag{16}$$

where the coefficients c_i are represented as;

$$c_i = \frac{1}{\left\|P_i^*(x)\right\|^2_{\omega_L}} \int_0^L u(x)\, P_i^*(x)\, \omega_L(x)\, dx, \qquad i = 0,1,2,..... \tag{17}$$

For m=2, we can obtain the following operational matrices.

D = $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix}$ D' = $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix}$

The method which was suggested is applied when m=2, a system of 3 linear algebraic equations follows, out of this 2 from the initial conditions and one from the main equation using the collocation point $x_0=0.5$ which is the root of $P_1^*(x) = 0$ which can be written in the matrix form:

$U(x) = P(x)\ A,$

where $P(x) = \left[P_0^*(x) \quad P_1^*(x) \quad P_2^*(x)\right] = \left[1 \quad 2x-1 \quad 6x^2 - 6x + 1\right],$

$A = \left[C_0 \quad C_1 \quad C_2\right]^T$

With the help of operational matrices, the given Eq.(9) together with the boundary conditions, we get the 3 algebraic equations and solve them by using Newton's method.

3.1 Fundamental relations

It is proposed the solution $u(x) \in C^m[0, L]$ can be estimated in terms of the first (m+1) terms of shifted Legendre polynomials given by

$$u(x) = \sum_{i=0}^{m} c_i\ P_i^*(x). \tag{18}$$

4. Method of solution by the LCM

The value of various parameters, useful for computing the solution, for the forehead region is taken as: $q = 0.000002T_p^2$; $T_p = 33.03 + 0.14(T_s - 10)$; $k_0 = 0.00009T_p(37-T_p)^{-1/3}$.

The value of T_s = 0°C, 10°C, 20°C, 25°C. Hence, each value of T_s is substituted in the equation of T_p and the solution is found. The matrices D and D' replace the first order and second order differential terms respectively in Eq. (9).

4.1 Limiting cases

Case 1: Consider T_s=0°C

The graph is obtained for the value of T_s=0°C. A comparison of results obtained by the FDM and LCM method is shown in Fig. 1. Experimental values from [17, 18] can be correlated to the ones obtained numerically.

Case 2: Then consider T_s=10°C

The graph is obtained for the value of T_s=10°C. A comparison of results obtained by the FDM and LCM method is shown in Fig. 2.

Case 3: Let us consider T_s=20°C

The graph is obtained for the value of T_s=20°C. A comparison of results obtained by the FDM and LCM method is shown in Fig. 3.

Case 4: Now we consider T_s = 25°C

The following graph is obtained for the value of T_s=25°C. A comparison of results obtained by the FDM and LCM method is shown in Fig. 4.

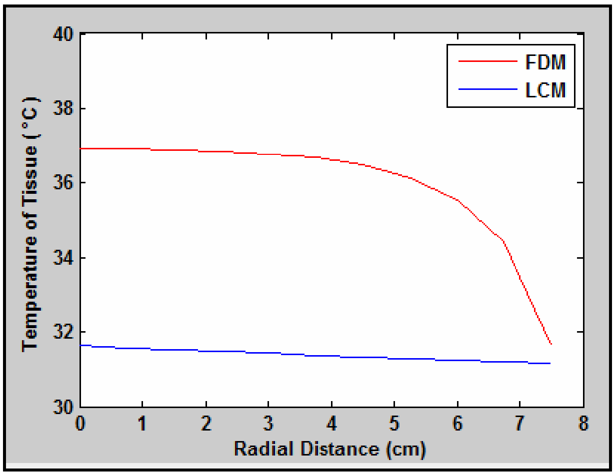

Fig.1 Comparison between FDM and LCM for T_s=0°C

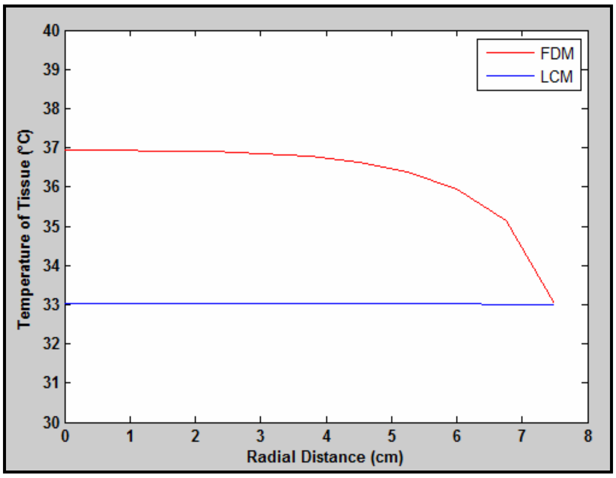

Fig.2 Comparison between FDM and LCM for T_s=10°C

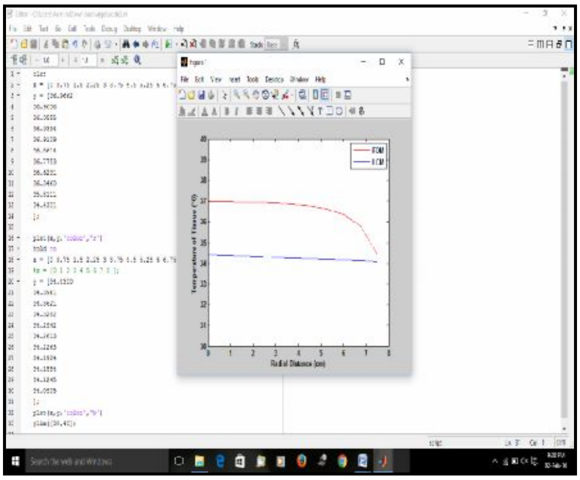

Fig.3 Comparison between FDM and LCM for T_s=20°C

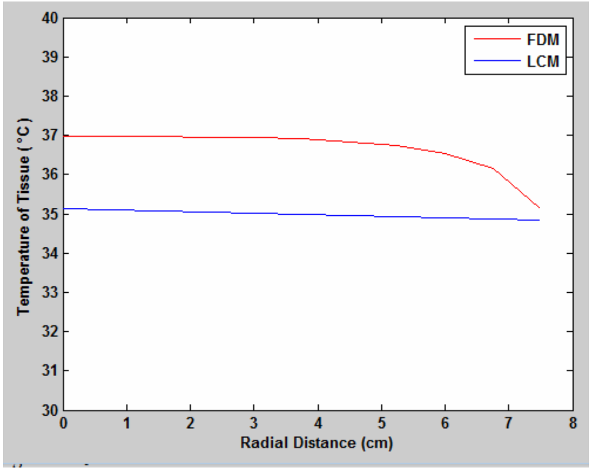

Fig.4 Comparison between FDM and LCM for T_s=25°C

The values plotted in the graphs above correlate with the experimental values. At an ambient temperature of around 25°C, the temperature of forehead was found to be around 35.2°C experimentally [18, 19].

5. Results and Discussion

In this paper, the Legendre computational matrix method is introduced to obtain the numerical solution singular nonlinear BVPs by governing the forehead region of the human body. The results are compared graphically with the results obtained by solving the equations with finite difference method. The various parameters are required for finding the solutions of the problem were adapted from [5, 20]. This study reflects an innovation in the technique that can be used to solve the singular nonlinear BVP pertaining to the estimation of thermoregulation in the dermal layers of the human body. The study in both FDM and LCM approach indicates that the thermal conductivity gradually decreases from the core to the outer regions as the temperature increases. This fact is immensely helpful for investigating the temperature variations of the human body especially during clinical or surgical situations.

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Nomenclature:

T_s	Surrounding atmosphere temperature
T_p	Peripheral temperature
R	Radius of human head
r_d	Radial temperature
c	Specific heat of tissue
ρ	Density of tissue
k	Thermal Conductivity
k_0	Initial thermal conductivity
E	Evaporation term
Q	Heat production per unit volume
q	Positive constant

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