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Lotka-Volterra Two-Species Mutualistic Biology Models and Their Ecological Monitoring

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Abstract: Lotka-Volterra population biology models are important models that describe the interaction between various biological species. Some important Lotka-Volterra population biology models are predator-prey models, competitive biology models and mutualistic biology models. In this research work, we study the two-species mutualistic biology models with carrying capacities and negative feedback. We show that for this two-species mutualistic biology model, under an assumption, the two mutualistic species have stable coexistence. Then we shall propose ecological monitoring of the two species cooperating population biology models by constructing nonlinear observers for them about their stable equilibrium points. The nonlinear observer design for the population biology model is constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the two-species mutualistic population biology model and the population size of any of the mutualistic species as the output function. Numerical examples are given to illustrate the ecological monitoring or the nonlinear observer design for the two-species cooperating biology model with stable coexistence. MATLAB simulations are shown to illustrate the numerical results shown in this research work.
Keywords: Population biology, Lotka-Volterra model, mutualism, cooperating system, two-species model, ecological monitoring, observer design, etc.

1. Introduction

Lotka-Volterra population biology models are important models that describe the interaction between various biological species considered as predator-prey system [1-2]. In the population biology literature, Lotka-Volterra two species competitive biology model is also well-known for the study of competing population models [3]. There are certain classes of biological models like the drosophila population models for which the dynamic behaviour cannot be sufficiently described by the Lotka-Volterra population biology models. Such competitive behaviour of population biology systems can be modeled by systems such as the Coleman-Gomatam logarithmic competitive biology model [4].

Mutualism is one of the major categories of the ecological interactions of the populations or species, where both mutualistic species can gain via the interactions [5]. Similar interactions within the same species are known as cooperation. A well-known example of mutualism is the relationship between ungulates and bacteria within their intestines. The ungulates benefit from the cellulase produced by the bacteria, which facilitates digestion and the bacteria benefit from having a stable supply of nutrients in the host environment. This can also be observed in many different symbiotic relationships. Mutualism plays a key part in ecology. In addition, mutualism is thought to have driven the evolution of much of the biological diversity we see, such as flower forms and co-evolution between groups of species [6].

We show that for the Lotka-Volterra two-species mutualistic biology model, under an assumption, the two mutualistic species have stable co-existence. After discussion on the Lotka-Volterra two species mutualistic biology models, we propose ecological monitoring of those biology models by explicitly constructing local exponential observers for the Lotka-Volterra two-species mutualistic biology models.

In control systems engineering, a *state observer* is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically computer-implemented, and provides the basis of many practical applications.

The problem of designing observers for linear control systems was first proposed and fully solved by Luenberger [7]. The problem of designing observers for nonlinear control systems was proposed by Thau [8]. Over the past three decades, significant attention has been paid in the control systems literature on the construction of observers for nonlinear control systems [9].

A characterization of local exponential observers for nonlinear control systems was first obtained by Sundarapandian [10]. In [10], necessary and sufficient conditions were obtained for exponential observers for Lyapunov stable continuous-time nonlinear systems. In [10], an exponential observer design was provided by Sundarapandian for nonlinear control systems, which generalizes the linear observer design of Luenberger [6] for linear control systems.

In [11], Sundarapandian obtained necessary and sufficient conditions for exponential observers for Lyapunov stable discrete-time nonlinear systems and also provided a formula for designing exponential observers for Lyapunov stable discrete-time nonlinear systems. In [12], Sundarapandian derived new results for the global observer design for nonlinear control systems.

The concept of nonlinear observers for nonlinear control systems was extended in many ways. In [13-14], Sundarapandian derived new results for the characterization of local exponential observers for nonlinear bifurcating systems. In [15-18], Sundarapandian derived new results for the exponential observer design for a general class of nonlinear systems with real parametric uncertainty. In [19-22], Sundarapandian derived new results for the general observers for nonlinear systems. In [23], Sundarapandian derived new results for observers around equilibria. In [24-25], Sundarapandian derived new results for the local observer design of periodic orbits of nonlinear systems.

In [26], Byrnes and Sundarapandian obtained a result for the persistence of equilibria for locally asymptotically stable systems. In [27], Sundarapandian derived a result for the stabilizability and a separation principle for periodic orbits. In [28], Sundarapandian derived a result for the global asymptotic stability of nonlinear cascade systems. In [29], Sundarapandian derived a necessary condition for local asymptotic stability of discrete-time nonlinear systems with parameters. In [30], Sundarapandian derived an invariance principle for discrete-time nonlinear systems.

In [31], Sundarapandian derived a relation between the output regulation and the observer design for nonlinear systems. In [32], Sundarapandian derived a necessary condition for local asymptotic stability of periodic orbits of nonlinear systems with parameters. In [33], Sundarapandian gave a geometric proof of Malkin's stability theorem. In [34], Sundarapandian gave sufficient conditions for the local and global asymptotic stability of nonlinear cascade interconnected systems. In [35], Sundarapandian derived distributed control schemes for large-scale interconnected discrete-time linear systems. In [36-37], Sundarapandian presented results for the observer design and stabilization of discrete-time linear systems. In [38-39], Sundarapandian derived a necessary condition for the local asymptotic stability of nonlinear systems with exogenous disturbance. In [40-41], Sundarapandian derived new results for the reduced order observer design for nonlinear systems.

2. Review of Nonlinear Observer Design for Nonlinear Systems

We consider the nonlinear system described by

$$\dot{x} = f(x) \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in R^n$ is the *state* and $y \in R^p$ is the *output*.

We assume that $f : R^n \to R^n$, $h : R^n \to R^p$ are C^1 mappings and for some $x^* \in R^n$, the following hold:

$$f(x^*) = 0, \quad h(x^*) = 0 \tag{2}$$

Remark 1. The solutions x^* of $f(x) = 0$ are called the *equilibrium points* of the system dynamics (1a). Also, the assumption $h(x^*) = 0$ holds without any loss of generality. Indeed, if $h(x^*) \neq 0$, then we can define a new output function as

$$\psi(x) = h(x) - h(x^*) \tag{3}$$

and it is easy to see that $\psi(x^*) = 0$. ■

The linearization of the nonlinear system (1a)-(1b) at $x = x^*$ is given by

$$\dot{x} = Ax \tag{4a}$$

$$y = Cx \tag{4b}$$

where

$$A = \left[\frac{\partial f}{\partial x} \right]_{x=x^*} \quad \text{and} \quad C = \left[\frac{\partial h}{\partial x} \right]_{x=x^*} \tag{5}$$

Definition 1. [23] A C^1 dynamical system defined by

$$\dot{z} = g(z, y), \quad (z \in R^n) \tag{6}$$

is called a **local asymptotic** (respectively, **local exponential**) observer for the nonlinear system (1a)-(1b) if the following two requirements are satisfied:

(O1) If $z(0) = x(0)$, then $z(t) = x(t)$, for all $t \geq 0$.

(O2) There exists a neighbourhood V of the equilibrium $x^* \in R^n$ such that for all $z(0), x(0) \in V$, the estimation error

$$e(t) = z(t) - x(t) \tag{7}$$

decays asymptotically (respectively, exponentially) to zero as $t \to \infty$. ■

Theorem 1. (Sundarapandian, [23]) Suppose that the nonlinear system dynamics (1a) is Lyapunov stable at the equilibrium $x = x^*$ and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the dynamical system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{8}$$

is a local exponential observer for the nonlinear system (1a)-(1b). ■

Remark 2. The estimation error is governed by the error dynamics

$$\dot{e} = f(x+e) - f(x) - K[h(x+e) - h(x)] \tag{9}$$

Linearizing the error dynamics (9) at $x = x^*$, we get the linear system

$$\dot{e} = Ee, \quad \text{where} \quad E = A - KC \tag{10}$$

If (C, A) is observable, then the eigenvalues of the error matrix $E = A - KC$ can be arbitrarily placed in the complex plane. Thus, when (C, A) is observable, a local exponential observer of the form (8) can be always found such that the transient response of the error decays quickly with any desired speed of convergence. ■

3. Lotka-Volterra Two Species Mutualistic Biology Models

In this section, we consider the Lotka-Volterra two-species mutualistic biology system [5], which is modeled by the system of differential equations

$$\begin{cases} \dot{x}_1 = x_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1}x_1 + \beta_1 x_2\right) \\ \dot{x}_2 = x_2\left(\alpha_2 + \beta_2 x_1 - \dfrac{\alpha_2}{K_2}x_2\right) \end{cases} \quad (11)$$

In (11), the two states x_1 and x_2 represent the population densities of the species 1 and 2, respectively. Also, $\alpha_1, \alpha_2, \beta_1, \beta_2, K_1, K_2$ are positive constants. We note that α_1, α_2 are the growth rate constants of the population species 1 and 2. K_1 and K_2 represent the carrying capacities of the species 1 and 2 respectively. The positive constants β_1 and β_2 represent the cooperation or mutualism in the ecological interactions of the two population species 1 and 2.

The equilibrium points of the system (11) are obtained by solving the system of equations

$$\begin{cases} x_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1}x_1 + \beta_1 x_2\right) = 0 \\ x_2\left(\alpha_2 + \beta_2 x_1 - \dfrac{\alpha_2}{K_2}x_2\right) = 0 \end{cases} \quad (12)$$

Clearly, there are three equilibrium points of the system (12) in the first quadrant given by

$$E_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} K_1 \\ 0 \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 \\ K_2 \end{bmatrix} \quad (13)$$

In this paper, we suppose that the mutualistic biology system (11) satisfies the following assumption:

(H) $\Delta = \alpha_1\alpha_2 - \beta_1\beta_2 K_1 K_2 > 0.$

Under the assumption (H), there is a fourth equilibrium point for the system (11) given by

$$E_4 = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}, \text{ where } x_1^* = \frac{\alpha_2 K_1}{\Delta}\left(\alpha_1 + \beta_1 K_2\right), \; x_2^* = \frac{\alpha_1 K_2}{\Delta}\left(\alpha_2 + \beta_2 K_1\right) \quad (14)$$

This can be seen as follows. If $\left(x_1^*, x_2^*\right)$ is a positive equilibrium of the system (11), then Eq. (12) yields

$$\begin{cases} \alpha_1 - \dfrac{\alpha_1}{K_1}x_1^* + \beta_1 x_2^* = 0 \\ \alpha_2 + \beta_2 x_1^* - \dfrac{\alpha_2}{K_2}x_2^* = 0 \end{cases} \quad (15)$$

which can be expressed in matrix form as

$$\begin{bmatrix} \dfrac{\alpha_1}{K_1} & -\beta_1 \\ -\beta_2 & \dfrac{\alpha_2}{K_2} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad (16)$$

We note that the coefficient matrix of the system (16) has a non-zero determinant by assumption (H). Thus, the system (16) has a unique solution.

Solving the system (16) by Cramer's rule, we get

$$x_1^* = \frac{\alpha_2 K_1}{\Delta}\left(\alpha_1 + \beta_1 K_2\right), \; x_2^* = \frac{\alpha_1 K_2}{\Delta}\left(\alpha_2 + \beta_2 K_1\right) \quad (17)$$

By assumption (H), $\Delta > 0$. Thus, $E_4 = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ is a positive equilibrium of the system (16).

The Jacobian or community matrix of the Lotka-Volterra system (11) at E_4 is obtained as

$$A = \begin{pmatrix} -\dfrac{\alpha_1 x_1^*}{K_1} & \beta_1 x_1^* \\ \beta_2 x_2^* & -\dfrac{\alpha_2 x_2^*}{K_2} \end{pmatrix} \tag{18}$$

Next, we find the characteristic equation of the community matrix A as

$$\lambda^2 - \text{Trace}(A)\lambda + \det(A) = 0 \tag{19}$$

We note that

$$\text{Trace}(A) = -\frac{\alpha_1 x_1^*}{K_1} - \frac{\alpha_2 x_2^*}{K_2} < 0 \tag{20}$$

and

$$\det(A) = \frac{\Delta x_1^* x_2^*}{K_1 K_2} > 0 \tag{21}$$

Since all the coefficients of the quadratic equation (22) are positive, it is immediate from Hurwitz criterion [42] that all the eigenvalues of the community matrix A are stable.
Thus, A is a Hurwitz matrix.

Thus, from Lyapunov stability theory [42], it is immediate that the positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ is locally asymptotically stable. Hence, we have proved the following theorem.

Theorem 2. Suppose that the Lotka-Volterra two-species mutualistic biology system (11) satisfies the assumption (H) stated as follows.

(H) $\Delta \quad \alpha_1\alpha_2 - \beta_1\beta_2 K_1 K_2 > 0.$

Then $E_4\left(x_1^*, x_2^*\right)$ is a positive equilibrium of the system (11). Also, the unique positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ of the Lotka-Volterra mutualistic biology system (11) is locally asymptotically stable. ■

4. Ecological Monitoring for the Lotka-Volterra Two-Species Mutualistic Biology Systems

In this section, we discuss how to do ecological monitoring of the Lotka-Volterra two-species mutualistic biology systems by designing a local exponential observer to estimate their states.

4.1 Ecological Monitoring of the Mutualistic Biology Models with Population Density of Species 1 as Output

We consider the Lotka-Volterra two-species mutualistic biology system given by

$$\begin{cases} \dot{x}_1 = x_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1}x_1 + \beta_1 x_2\right) \\ \dot{x}_2 = x_2\left(\alpha_2 + \beta_2 x_1 - \dfrac{\alpha_2}{K_2}x_2\right) \end{cases} \tag{22}$$

We suppose that the population density of species 1 is given as the system output, *i.e.*

$$y = x_1 \tag{23}$$

We suppose that the assumption (H) holds so that $\left(x_1^*, x_2^*\right)$ is a unique positive equilibrium of the system (22).

In Section 3, we showed that the community matrix of the system (22) about the unique positive equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$A = \begin{pmatrix} -\dfrac{\alpha_1 x_1^*}{K_1} & \beta_1 x_1^* \\ \beta_2 x_2^* & -\dfrac{\alpha_2 x_2^*}{K_2} \end{pmatrix} \tag{24}$$

which is a Hurwitz matrix. Thus, the equilibrium $\left(x_1^*, x_2^*\right)$ is locally asymptotically stable.

Moreover, the linearization of the output function (23) about the equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{25}$$

Thus, the observability matrix for the system (22)-(23) is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{\alpha_1 x_1^*}{K_1} & \beta_1 x_1^* \end{bmatrix} \tag{26}$$

We find that

$$`\det(W) = \beta_1 x_1^* \neq 0 \tag{27}$$

which shows that the observability matrix W has full rank.
Thus, by Kalman's rank test for observability [43], the system (22)-(23) is completely observable.

Hence, by Sundarapandian's theorem (Theorem 1, Section 2), we obtain the following main result, which gives the ecological monitoring of the Lotka-Volterra two-species mutualistic biology systems.

Theorem 3. Suppose that the assumption (H) is satisfied. Then the Lotka-Volterra two-species mutualistic biology system (22) with output (23) has a local exponential observer of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1} z_1 + \beta_1 z_2\right) \\ z_2\left(\alpha_2 + \beta_2 z_1 - \dfrac{\alpha_2}{K_2} z_1\right) \end{bmatrix} + L\left[y - z_1\right] \tag{28}$$

where L is a matrix chosen such that $A - LC$ is Hurwitz. Since (C, A) is observable, an observer gain matrix L can be found such that the error matrix $E = A - LC$ has arbitrarily assigned set of stable eigenvalues. ■

Example 1. We consider a Lotka-Volterra two species mutualistic biology system given by

$$\begin{cases} \dot{x}_1 = x_1\left(2 - x_1 + x_2\right) \\ \dot{x}_2 = x_2\left(4 + x_1 - 2x_2\right) \end{cases} \tag{29}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.
Suppose that the output function given by the density of mutualistic species 1, *i.e.*

$$y = x_1 \tag{30}$$

Here,

$$\alpha_1 = 2, \ \alpha_2 = 4, \ K_1 = 2, \ K_2 = 2, \ \beta_1 = 1, \ \beta_2 = 1 \tag{31}$$

We note that

$$\Delta = \alpha_1\alpha_2 - \beta_1\beta_2 K_1 K_2 = 4 > 0 \tag{32}$$

Thus, the assumption (H) is satisfied.
We find the positive equilibrium of the system (29) by solving the equations

$$\begin{cases} x_1\left(2 - x_1 + x_2\right) = 0 \\ x_2\left(4 + x_1 - 2x_2\right) = 0 \end{cases} \tag{33}$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we obtain

$$\begin{cases} 2 - x_1 + x_2 = 0 \\ 4 + x_1 - 2x_2 = 0 \end{cases} \tag{34}$$

Then the system (34) can be easily arranged in matrix form as

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \tag{35}$$

By solving the linear system (35), we get the unique solution as

$$x_1^* = 8, \ \ x_2^* = 6 \tag{36}$$

which is the positive equilibrium point of the Lotka-Volterra two-species mutualistic biology system (29).
As shown in Section 3, the Lotka-Volterra two-species mutualistic biology system (29) is locally asymptotically stable about the unique positive equilibrium point (x_1^*, x_2^*).

The linearization of the two-species mutualistic biology dynamics (29) at (x_1^*, x_2^*) is given by

$$A = \begin{bmatrix} -8 & -8 \\ 6 & -12 \end{bmatrix} \tag{37}$$

Since $\text{Trace}(A) = -20 < 0$ and $\det(A) = 144 > 0$, it is evident that A is a Hurwitz matrix.

Also, the linearization of the output function (29) at (x_1^*, x_2^*) is given by

$$C = [1 \quad 0] \tag{38}$$

It is easy to check that the observability matrix $W = \begin{bmatrix} C \\ CA \end{bmatrix}$ has full rank.

This shows that the given system (29) with output (30) is completely observable near the positive equilibrium point $(x_1^*, x_2^*) = (8, 6)$.

For numerical simulations, we take $x_1(0) = 2.5$ and $x_2(0) = 15.8$.

Figure 1 illustrates that the unique positive equilibrium point $(x_1^*, x_2^*) = (8, \ 6)$ is locally asymptotically stable.

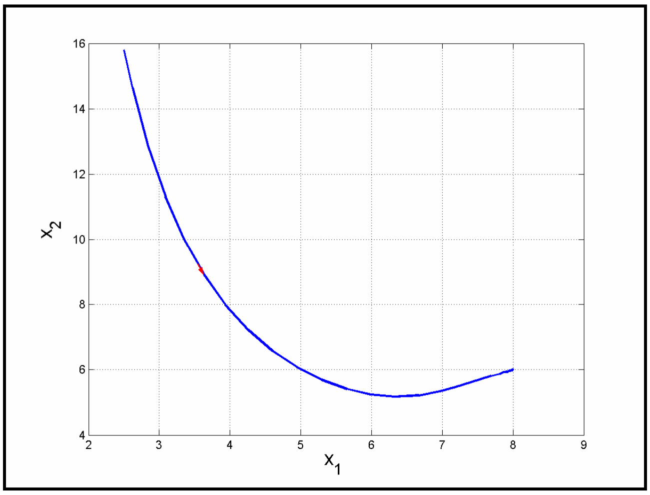

Figure 1. State Orbit of the Two-Species Mutualistic Biology System (32)

Since (C, A) is observable, the eigenvalues of the error matrix $E = A - LC$ can be placed arbitrarily.

Using the Ackermann's formula [23] for the observer gain matrix, we can choose L so that the error matrix $E = A - LC$ has the stable eigenvalues $\{-8, -8\}$.

A simple calculation using MATLAB gives

$$L = \begin{bmatrix} -4 \\ 4 \end{bmatrix}. \tag{39}$$

By Theorem 3, a local exponential observer for the Lotka-Volterra two-species mutualistic system (29)-(30) around the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (60,\ 80)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(2 - z_1 + z_2) \\ z_2(4 + z_1 - 2z_2) \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} [y - z_1] \tag{40}$$

For simulations, we choose the initial conditions of the plant dynamics (29) as

$$x_1(0) = 12, \quad x_2(0) = 5 \tag{41}$$

Also, we choose the initial conditions of the observer dynamics (40) as

$$z_1(0) = 4, \quad z_2(0) = 18 \tag{42}$$

Figures 2-3 depict the exponential convergence of the observer states z_1 and z_2 of the system (40) to the states x_1 and x_2 of the Lotka-Volterra two-species mutualistic biology system (29)-(30). Figure 4 shows the time-history of the estimation errors e_1, e_2, where $e_1 = z_1 - x_1$ and $e_2 = z_2 - x_2$.

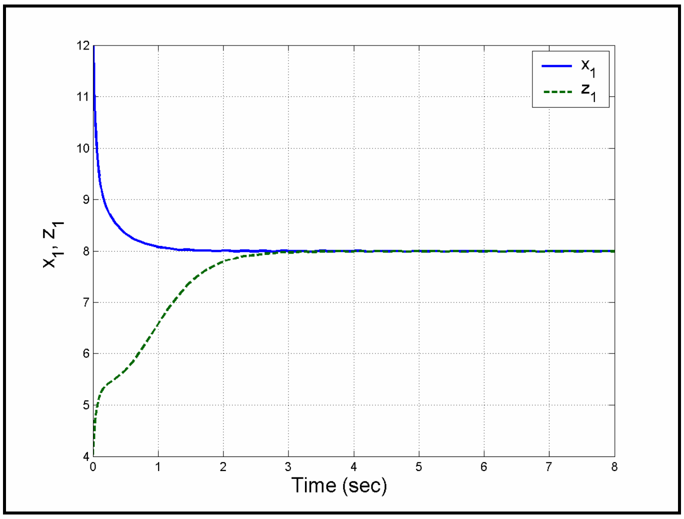

Figure 2. Synchronization of the states x_1 and z_1

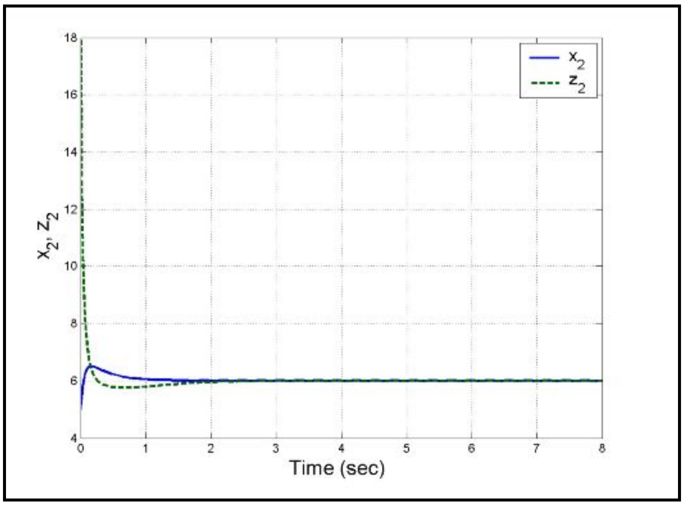

Figure 3. Synchronization of the states x_2 and z_2

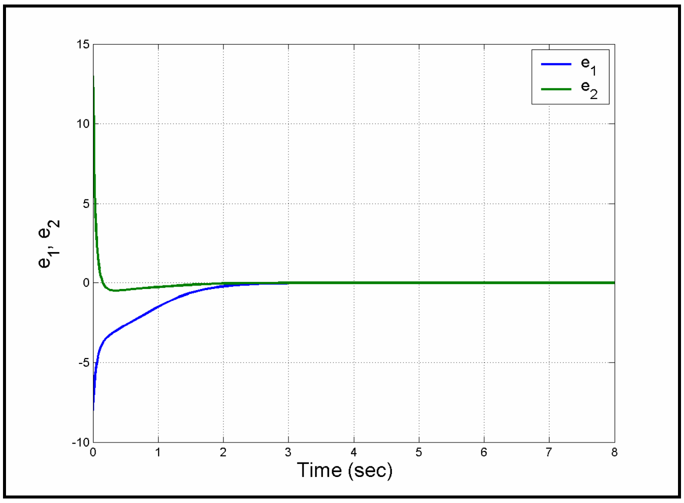

Figure 4. Time-history of the estimation errors e_1 and e_2

4.2 Ecological Monitoring of the Mutualistic Models with Population Density of Species 2 as Output

We consider the Lotka-Volterra two-species mutualistic biology system given by

$$\begin{cases} \dot{x}_1 = x_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1}x_1 + \beta_1 x_2\right) \\ \dot{x}_2 = x_2\left(\alpha_2 + \beta_2 x_1 - \dfrac{\alpha_2}{K_2}x_2\right) \end{cases} \tag{43}$$

We suppose that the population density of species 2 is given as the system output, *i.e.*

$$y = x_2 \tag{44}$$

We suppose that the assumption (H) holds so that (x_1^*, x_2^*) is a unique positive equilibrium of the system (43).

In Section 3, we showed that the community matrix of the system (43) about the unique positive equilibrium (x_1^*, x_2^*) is given by

$$A = \begin{pmatrix} -\dfrac{\alpha_1 x_1^*}{K_1} & \beta_1 x_1^* \\ \beta_2 x_2^* & -\dfrac{\alpha_2 x_2^*}{K_2} \end{pmatrix}, \tag{45}$$

which is a Hurwitz matrix. Thus, the equilibrium (x_1^*, x_2^*) is locally asymptotically stable.

Moreover, the linearization of the output function (44) about the equilibrium (x_1^*, x_2^*) is given by

$$C = [0 \quad 1] \tag{46}$$

Thus, the observability matrix for the system (43)-(44) is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \beta_2 x_2^* & -\dfrac{\alpha_2 x_2^*}{K_2} \end{bmatrix} \tag{47}$$

We find that

` $\det(W) = -\beta_2 x_2^* \neq 0,$ (48)

which shows that the observability matrix W has full rank.

Thus, by Kalman's rank test for observability [26], the system (43)-(44) is completely observable.

Hence, by Sundarapandian's theorem (Theorem 1, Section 2), we obtain the following main result, which gives the ecological monitoring of the Lotka-Volterra two-species mutualistic biology systems.

Theorem 4. Suppose that the assumption (H) is satisfied. Then the Lotka-Volterra two-species mutualistic biology system (43) with output (44) has a local exponential observer of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1\left(\alpha_1 - \dfrac{\alpha_1}{K_1} z_1 + \beta_1 z_2\right) \\ z_2\left(\alpha_2 + \beta_2 z_1 - \dfrac{\alpha_2}{K_2} z_1\right) \end{bmatrix} + L\left[y - z_1\right] \quad (49)$$

where L is a matrix chosen such that $A - LC$ is Hurwitz. Since (C, A) is observable, an observer gain matrix L can be found such that the error matrix $A - LC$ has arbitrarily assigned set of stable eigenvalues. ■

Example 2. We consider a Lotka-Volterra two-species mutualistic biology system given by

$$\begin{cases} \dot{x}_1 = x_1\left(4 - 2x_1 + x_2\right) \\ \dot{x}_2 = x_2\left(2 + x_1 - x_2\right) \end{cases} \quad (50)$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Suppose that the output function given by the density of competing species 2, *i.e.*

$$y = x_2 \quad (51)$$

Here,

$$(52)$$

We note that

$$\Delta \quad \alpha_1\alpha_2 - \beta_1\beta_2 K_1 K_2 \quad 4 > 0. \quad (53)$$

Thus, the assumption (H) is satisfied.

We find the positive equilibrium of the system (55) by solving the equations

$$\begin{cases} x_1\left(4 - 2x_1 + x_2\right) = 0 \\ x_2\left(2 + x_1 - x_2\right) = 0 \end{cases} \quad (54)$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we obtain

$$\begin{cases} 4 - 2x_1 + x_2 = 0 \\ 2 + x_1 - x_2 = 0 \end{cases} \quad (55)$$

Then the system (55) can be easily arranged in matrix form as

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad (56)$$

By solving the linear system (56), we get the unique solution as

$$x_1^* = 6, \;\; x_2^* = 8 \quad (57)$$

As shown in Section 3, the Lotka-Volterra two-species mutualistic biology system (55) is locally asymptotically stable about the unique positive equilibrium point $\left(x_1^*, x_2^*\right)$.

The linearization of the Lotka-Volterra two-species mutualistic biology dynamics (49) at $\left(x_1^*, x_2^*\right)$ is given by

$$A = \begin{bmatrix} -12 & 6 \\ 8 & -8 \end{bmatrix} \quad (58)$$

Also, the linearization of the output function (56) at $\left(x_1^*, x_2^*\right)$ is given by

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (59)$$

It is easy to check that the observability matrix $W = \begin{bmatrix} C \\ CA \end{bmatrix}$ has full rank. This shows that the given system (55) with output (56) is completely observable near the positive equilibrium point $(x_1^*, x_2^*) = (6,8)$.

For numerical simulations, we take $x_1(0) = 15.7$ and $x_2(0) = 2.5$.

Figure 5 illustrates that the unique positive equilibrium point $(x_1^*, x_2^*) = (6,8)$ is locally asymptotically stable.

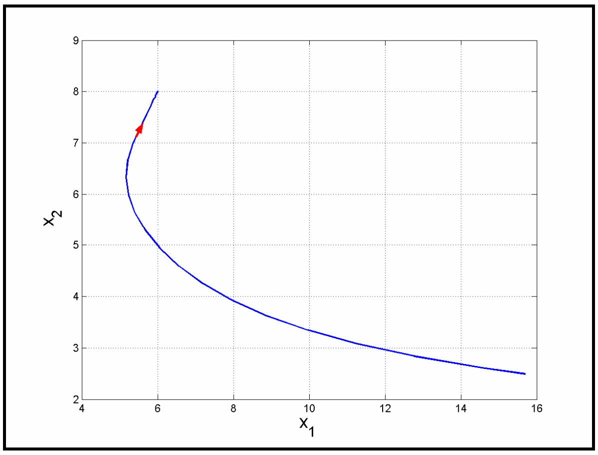

Figure 5. State orbit of the two-species mutualistic biology system (50)

Since (C, A) is observable, the eigenvalues of the error matrix $E = A - LC$ can be placed arbitrarily.
Using the Ackermann's formula [23] for the observer gain matrix, we can choose L so that the error matrix $E = A - LC$ has the stable eigenvalues $\{-8, -8\}$.
A simple calculation using MATLAB gives

$$L = \begin{bmatrix} 8 \\ -4 \end{bmatrix}. \tag{60}$$

By Theorem 4, a local exponential observer for the two-species mutualistic biology system (50)-(51) around the unique positive equilibrium point $(x_1^*, x_2^*) = (6,8)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(4 - 2z_1 + z_2) \\ z_2(2 + z_1 - z_2) \end{bmatrix} + \begin{bmatrix} 8 \\ -4 \end{bmatrix} [y - z_2] \tag{61}$$

For simulations, we choose the initial conditions of the plant dynamics (55) as

$$x_1(0) = 12, \quad x_2(0) = 5 \tag{62}$$

Also, we choose the initial conditions of the observer dynamics (68) as

$$z_1(0) = 7, \quad z_2(0) = 9 \tag{63}$$

Figures 6-7 depict the exponential convergence of the observer states z_1 and z_2 of the system (61) to the states x_1 and x_2 of the two-species mutualistic biology system (50)-(51). Figure 8 shows the time-history of the estimation errors e_1, e_2, where $e_1 = z_1 - x_1$ and $e_2 = z_2 - x_2$.

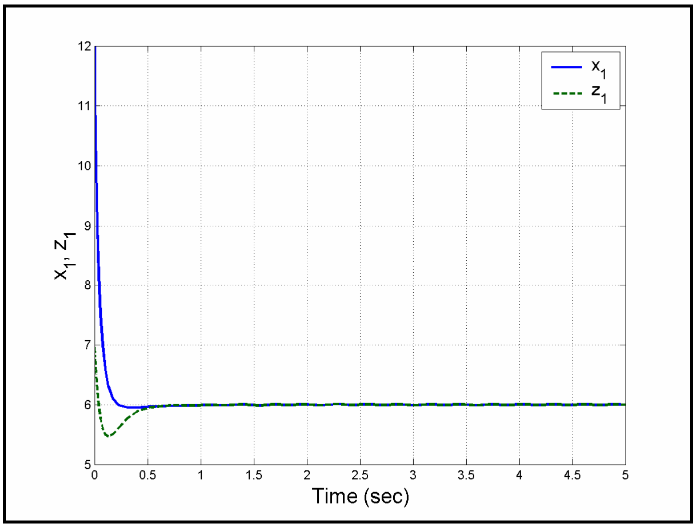

Figure 6. Synchronization of the states x_1 and z_1

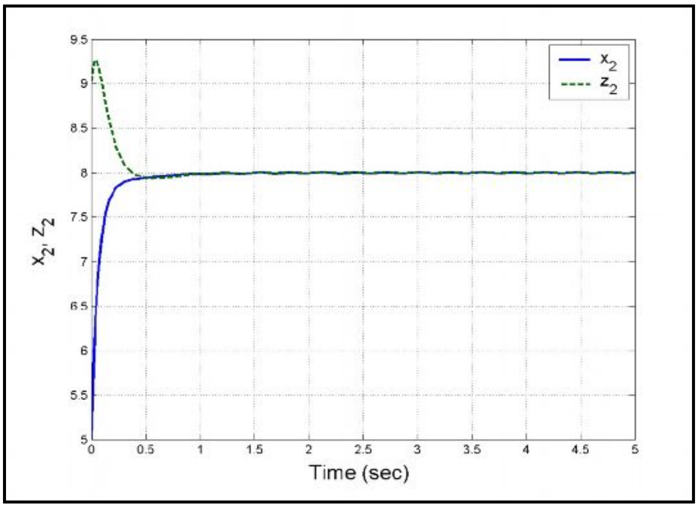

Figure 7. Synchronization of the states x_2 and z_2

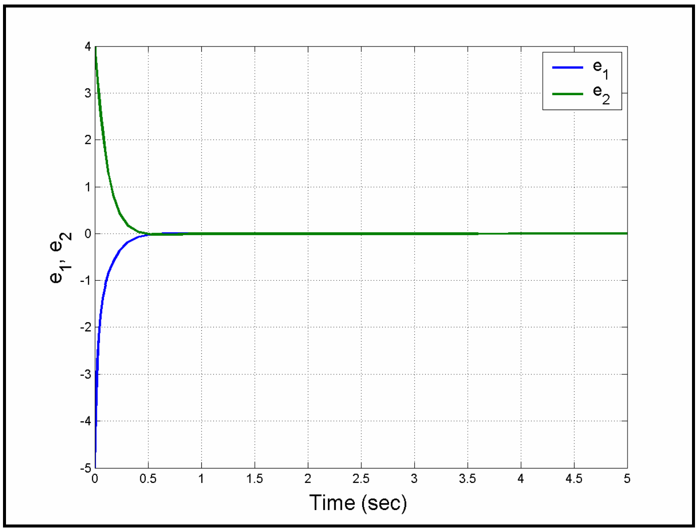

Figure 8. Time-history of the estimation errors e_1 and e_2

5. Conclusions

In this paper, we described Lotka-Volterra two-species mutualistic biology models. We showed that for this biological model, under an assumption, the two mutualistic species have stable coexistence. Then we achieved ecological monitoring of the population biology model by constructing a nonlinear exponential observer for the mutualistic biology model under study. The nonlinear observer design for the mutualistic biology model was constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the Lotka-Volterra two-species mutualistic population biology model and any of the density of the two mutualistic species as the output function. Numerical example and MATLAB simulations were shown to illustrate the ecological monitoring or the nonlinear observer design for the Lotka-Volterra two-species mutualistic biology models.

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