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Lotka-Volterra Population Biology Models with Negative Feedback and their Ecological Monitoring

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Abstract : Lotka-Volterra population biology models are important models that describe the interaction between various biological species considered as predator-prey system. This work describes a Lotka-Volterra population biology model with negative feedback. We show that for this biological model, the predator and prey species have stable coexistence. Then we shall propose ecological monitoring of the population biology model by constructing a nonlinear exponential observer for the population biology model under study. The nonlinear observer design for the population biology model is constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the Lotka-Volterra population biology model and the prey population size as the output function. Numerical example and MATLAB simulations are given to illustrate the ecological monitoring or the nonlinear observer design for the two-species Lotka-Volterra population biology model with negative feedback.
Keywords: Population biology, Lotka-Volterra model, ecological monitoring, observer design, etc.

1. Introduction

Lotka-Volterra population biology models are important models that describe the interaction between various biological species considered as predator-prey system [1-2]. This work describes a Lotka-Volterra population biology model with negative feedback. We show that for this biological model, the predator and prey species have stable coexistence. After discussion on the Lotka-Volterra population biology models, we propose ecological monitoring of the Lotka-Volterra predator-prey population biology model by explicitly constructing a nonlinear exponential observer for the population biology model.

The problem of designing observers for linear control systems was first proposed and fully solved by Luenberger [3]. The problem of designing observers for nonlinear control systems was proposed by Thau [4]. Over the past three decades, significant attention has been paid in the control systems literature on the construction of observers for nonlinear control systems [5].

A characterization of local exponential observers for nonlinear control systems was first obtained by Sundarapandian [6]. In [6], necessary and sufficient conditions were obtained for exponential observers for Lyapunov stable continuous-time nonlinear systems. In [6], an exponential observer design was provided by Sundarapandian for nonlinear control systems, which generalizes the linear observer design of Luenberger [3] for linear control systems. In [7], Sundarapandian obtained necessary and sufficient conditions for exponential observers for Lyapunov stable discrete-time nonlinear systems and also provided a formula for designing exponential observers for Lyapunov stable discrete-time nonlinear systems. In [8], Sundarapandian derived new results for the global observer design for nonlinear control systems.

The concept of nonlinear observers for nonlinear control systems was extended in many ways. In [9-

10], Sundarapandian derived new results for the characterization of local exponential observers for nonlinear bifurcating systems. In [11-14], Sundarapandian derived new results for the exponential observer design for a general class of nonlinear systems with real parametric uncertainty. In [15-18], Sundarapandian derived new results for the general observers for nonlinear systems. In [19], Sundarapandian derived new results for observers around equilibria.

This work discusses the Lotka-Volterra population biology model with negative feedback. Section 2 reviews the definition and results of local exponential observers for nonlinear systems. Section 3 describes the Lotka-Volterra population biology model with negative feedback. In this section, we show that for this biological model, the predator and prey species have stable coexistence. This stability result for the population biology model is established by applying Lyapunov stability theory [20]. Section 4 details the ecological monitoring of the population biology model discussed in Section 3 by giving a design of nonlinear exponential observer that estimates the states of the population biology model. Section 5 contains the conclusions of this work.

2. Review of Nonlinear Observer Design for Nonlinear Systems

Observers for nonlinear systems help to estimate the states of the nonlinear systems. Mathematically, observers for nonlinear systems in the neighbourhood of an equilibrium point are defined as in [19].

Mathematically, observers for nonlinear systems are defined as follows.

We consider the nonlinear system described by

$$\dot{x} = f(x) \tag{1a}$$

$$y = h(x) \tag{1b}$$

where $x \in R^n$ is the *state* and $y \in R^p$ is the *output*.

We assume that $f : R^n \to R^n$, $h : R^n \to R^p$ are C^1 mappings and for some $x^* \in R^n$, the following hold:

$$f(x^*) = 0, \quad h(x^*) = 0 \tag{2}$$

Remark 1. The solutions x^* of $f(x) = 0$ are called the *equilibrium points* of the system dynamics (1a). Also, the assumption $h(x^*) = 0$ holds without any loss of generality. Indeed, if $h(x^*) \neq 0$, then we can define a new output function as

$$\psi(x) = h(x) - h(x^*) \tag{3}$$

and it is easy to see that $\psi(x^*) = 0$. ■

The linearization of the nonlinear system (1a)-(1b) at $x = x^*$ is given by

$$\dot{x} = Ax \tag{4a}$$

$$y = Cx \tag{4b}$$

where

$$A = \left[\frac{\partial f}{\partial x} \right]_{x=x^*} \quad \text{and} \quad C = \left[\frac{\partial h}{\partial x} \right]_{x=x^*} \tag{5}$$

Next, we state the fundamental definition for the local asymptotic or exponential observers for nonlinear dynamical systems. Essentially, an observer for a nonlinear system is a state estimator, and the states of the observer converge to the states of the plant dynamics asymptotically or exponentially as time tends to infinity.

Definition 1. [19] A C^1 dynamical system defined by

$$\dot{z} = g(z, y), \quad (z \in R^n) \tag{6}$$

is called a **local asymptotic** (respectively, **local exponential**) observer for the nonlinear system (1a)-(1b) if the following two requirements are satisfied:

(O1) If $z(0) = x(0)$, then $z(t) = x(t)$, for all $t \geq 0$.

(O2) There exists a neighbourhood V of the equilibrium $x^* \in R^n$ such that for all $z(0), x(0) \in V$, the estimation error

$$e(t) = z(t) - x(t) \tag{7}$$

decays asymptotically (respectively, exponentially) to zero as $t \to \infty$. ■

Theorem 1. (Sundarapandian, [19]) Suppose that the nonlinear system dynamics (1a) is Lyapunov stable at the equilibrium $x = x^*$ and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the dynamical system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{8}$$

is a local exponential observer for the nonlinear system (1a)-(1b). ■

Remark 2. The estimation error is governed by the error dynamics

$$\dot{e} = f(x+e) - f(x) - K[h(x+e) - h(x)] \tag{9}$$

Linearizing the error dynamics (9) at $x = x^*$, we get the linear system

$$\dot{e} = Ee, \qquad \text{where} \quad E = A - KC \tag{10}$$

If (C, A) is observable, then the eigenvalues of the error matrix $E = A - KC$ can be arbitrarily placed in the complex plane. Thus, when (C, A) is observable, a local exponential observer of the form (8) can be always found such that the transient response of the error decays quickly with any desired speed of convergence. ■

3. Lotka-Volterra Population Biology Systems with Negative Feedback

In this section, we consider a Lotka-Volterra predator-prey population biology system with negative feedback, which is modeled by the system of differential equations

$$\begin{cases} \dot{x}_1 = ax_1 - bx_1^2 - cx_1x_2 \\ \dot{x}_2 = -dx_2 + \varepsilon x_1x_2 - fx_2^2 \end{cases} \tag{11}$$

where x_1 denotes the number of preys, x_2 denotes the number of predators and $a, b, c, d, \varepsilon, f$ are positive constants.

In (11), the quadratic nonlinear terms $-bx_1^2$ and $-fx_2^2$ represent the negative feedback.

It is easy to see that the Lotka-Volterra population biology system (11) has four equilibria:

$$E_1(0,0), \ E_2\left(\frac{a}{b}, 0\right), \ E_3\left(0, -\frac{d}{f}\right), \ E_4\left(x_1^*, x_2^*\right) \tag{12}$$

Where

$$x_1^* = \frac{af + cd}{bf + c\varepsilon}, \ x_2^* = \frac{a\varepsilon - bd}{bf + c\varepsilon} \tag{13}$$

Of the four equilibria E_1, E_2, E_3, E_4, only the equilibrium point $E_4\left(x_1^*, x_2^*\right)$ has positive coordinates provided the following inequality holds:

$$a\varepsilon - bd > 0 \ \text{ or } \ \frac{a}{d} > \frac{b}{\varepsilon} \tag{14}$$

The Jacobian or community matrix corresponding to $E_4(x_1^*, x_2^*)$ is obtained as

$$A = \begin{pmatrix} -bx_1^* & -cx_1^* \\ \varepsilon x_2^* & -fx_2^* \end{pmatrix} \tag{15}$$

Next, we find the characteristic equation of the community matrix A as

$$\lambda^2 + (bx_1^* + fx_2^*)\lambda + (bf + c\varepsilon)x_1^*x_2^* = 0 \tag{16}$$

Since all the coefficients of the quadratic equation (16) are positive, it is immediate from Hurwitz criterion that all the eigenvalues of the community matrix A are stable. Thus, A is a Hurwitz matrix.

Thus, from Lyapunov stability theory [20], it is immediate that the positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ is locally asymptotically stable.

Next, we will prove a theorem giving a region of asymptotic stability for $E_4\left(x_1^*, x_2^*\right)$.

Theorem 2. We suppose that the inequality (14) holds so that $E_4\left(x_1^*, x_2^*\right)$ is a positive equilibrium of the Lotka-Volterra population biology system (11). The unique positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ of the Lotka-Volterra population biology system (11) is asymptotically stable in the region

$$\Omega \quad \left\{(y_1, y_2) \in R^2 : y_1 + x_1^* > 0,\ y_2 + x_2^* > 0\right\}.$$

Proof. First, we make a change of coordinates

$$\begin{cases} x_1 = y_1 + x_1^* \\ x_2 = y_2 + x_2^* \end{cases} \tag{17}$$

Under the coordinates transformation (17), the equilibrium $E_4\left(x_1^*, x_2^*\right)$ in the $(x_1, x_2)-$plane gets mapped onto $E_4(0,0)$ in the $(y_1, y_2)-$plane.

In the new $y-$coordinates, the Lotka-Volterra population biology system (11) becomes

$$\begin{cases} \dot{y}_1 = (y_1 + x_1^*)(-by_1 - cy_2) \\ \dot{y}_2 = (y_2 + x_2^*)(\varepsilon y_1 - fy_2) \end{cases} \tag{18}$$

Next, we investigate the stability of the planar system (18) by considering the scalar function

$$V(y_1, y_2) = \int_0^{y_1} \frac{\varepsilon\tau}{\tau + x_1^*}\, d\tau + \int_0^{y_2} \frac{c\tau}{\tau + x_2^*}\, d\tau \tag{19}$$

It is clear that $V(0,0) = 0$ and $V(y_1, y_2) > 0$ in the region

$$\Omega \quad \left\{(y_1, y_2) \in R^2 : y_1 + x_1^* > 0,\ y_2 + x_2^* > 0\right\}.$$

Thus, V is a positive definite function in the region Ω.

Next, the time-derivative of V along the solutions of the Lotka-Volterra population biology system (18) is obtained as

$$\frac{dV}{dt} = \frac{\varepsilon y_1 \dot{y}_1}{y_1 + x_1^*} + \frac{c y_2 \dot{y}_2}{y_2 + x_2^*} = -b\varepsilon y_1^2 - cfy_2^2 \tag{20}$$

Thus, $\dfrac{dV}{dt}$ is negative definite in the region Ω.

Hence, by Lyapunov stability theory [20], the equilibrium $E_4\left(x_1^*, x_2^*\right)$ is asymptotically stable in the region Ω.

This completes the proof. ■

Remark 3. The Lotka-Volterra two-species population biology system with negative feedback described by (11) is realistic in the following sense. When there is no predator ($x_2 = 0$), the prey population increases and eventually stabilizes at its *carrying capacity* $\dfrac{a}{b}$. When there is no prey ($x_1 = 0$), the predator population decreases for lack of food and slowly becomes *extinct*.

However, when both prey and predator populations have negative feedbacks, their populations are in *stable coexistence*.

The condition $\dfrac{a}{d} > \dfrac{b}{\varepsilon}$ given by (14) is necessary for the equilibrium state $E_4(x_1^*, x_2^*)$ to be positive. The Lotka-Volterra two-species population biology system with negative feedback described by (11) is hence an important model in population biology. ■

4. Ecological Monitoring for the Lotka-Volterra Two-Species Population Biology Systems with Negative Feedback

In this section, we discuss how to do ecological monitoring of the Lotka-Volterra two-species population biology systems with negative feedback by designing a local exponential observer to estimate their states.

We consider the Lotka-Volterra two-species population biology system with negative feedback given by

$$\begin{cases} \dot{x}_1 = ax_1 - bx_1^2 - cx_1x_2 \\ \dot{x}_2 = -dx_2 + \varepsilon x_1x_2 - fx_2^2 \end{cases} \tag{21}$$

We suppose that the prey population is given as the output function for the Lotka-Volterra population biology system with negative feedback, *i.e.*

$$y = x_1 \tag{22}$$

We suppose that the inequality (14) holds so that (x_1^*, x_2^*) is a unique positive equilibrium of the system (21), where

$$x_1^* = \frac{af + cd}{bf + c\varepsilon}, \quad x_2^* = \frac{a\varepsilon - bd}{bf + c\varepsilon} \tag{23}$$

In Section 3, we showed that the community matrix of the system (21) about the unique positive equilibrium (x_1^*, x_2^*) is given by

$$A = \begin{pmatrix} -bx_1^* & -cx_1^* \\ \varepsilon x_2^* & -fx_2^* \end{pmatrix} \tag{24}$$

is a Hurwitz matrix. Thus, the equilibrium (x_1^*, x_2^*) is locally asymptotically stable.

Moreover, the linearization of the output function (22) about the equilibrium (x_1^*, x_2^*) is given by

$$C = [1 \quad 0] \tag{25}$$

Thus, the observability matrix for the system (21)-(22) is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -bx_1^* & -cx_1^* \end{bmatrix} \tag{26}$$

We find that $|W| = -cx_1^* \neq 0$, which shows that the observability matrix W has full rank.

Thus, by Kalman's rank test for observability [21], the system (21)-(22) is completely observable.

Hence, by Sundarapandian's theorem (Theorem 1, Section 2), we obtain the following main result, which gives the ecological monitoring of the Lotka-Volterra population biology systems with negative feedback.

Theorem 3. The Lotka-Volterra population biology system with negative feedback (21) with output (22) has a local exponential observer of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} az_1 - bz_1^2 - cz_1z_2 \\ -dz_2 + \varepsilon z_1z_2 - fz_2^2 \end{bmatrix} + K[y - z_1] \tag{27}$$

where K is a matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, an observer gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of stable eigenvalues. ■

5. Numerical Example

We consider a two species Lotka-Volterra population biology system with negative feedback given by

$$\begin{cases} \dot{x}_1 = x_1(4 - 2x_1 - x_2) \\ \dot{x}_2 = x_2(-2 + 2x_1 - x_2) \end{cases} \tag{28}$$

with the output function given by the prey population

$$y = x_1 \tag{29}$$

We find the positive equilibrium of the system (28) by solving the equations

$$\begin{cases} x_1(4 - 2x_1 - x_2) = 0 \\ x_2(-2 + 2x_1 - x_2) = 0 \end{cases} \tag{30}$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we obtain

$$4 - 2x_1 - x_2 = 0 \ \text{ and } \ -2 + 2x_1 - x_2 = 0 \tag{31}$$

This can be easily arranged in matrix form as

$$\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{32}$$

By solving the system (32), we get the unique positive equilibrium point as $\left(x_1^*, x_2^*\right) = (1.5,\ 1)$.

As shown in Section 2, the Lotka-Volterra population biology system (28) is locally asymptotically stable aobut the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (1.5,\ 1)$.

For numerical simulations, we take $x_1(0) = 0.8$ and $x_2(0) = 2.4$. Figure 1 depicts the corresponding phase portrait of the Lotka-Volterra population biology system (28). Figure 1 illustrates that the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (1.5,\ 1)$ is locally asymptotically stable.

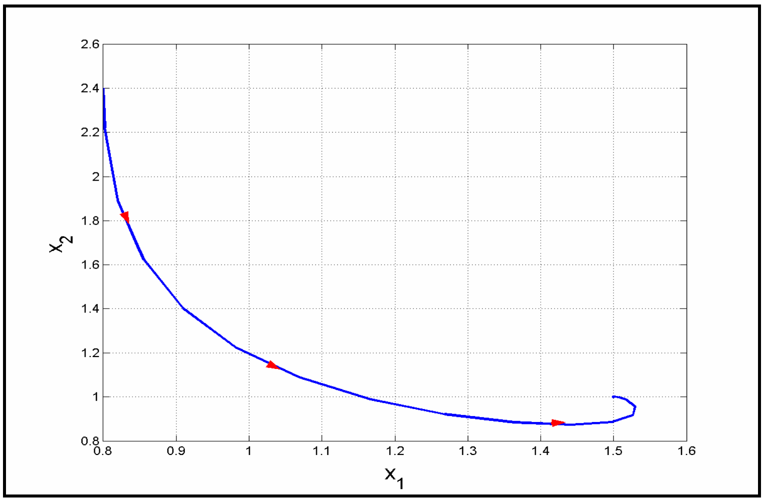

Figure 1. State Orbit of the Lotka-Volterra Population Biology System

The linearization of the population biology dynamics (28) at $\left(x_1^*, x_2^*\right) = (1.5, 1)$ is given by

$$A = \begin{bmatrix} -3 & -1.5 \\ 2 & -1 \end{bmatrix} \tag{33}$$

Also, the linearization of the output function (29) at $\left(x_1^*, x_2^*\right) = (1.5, 1)$ is given by

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{34}$$

Since (C, A) is observable, the eigenvalues of the error matrix $E = A - KC$ can be placed arbitrarily.

Using the Ackermann's formula [21] for the observer gain matrix, we can choose K so that the error matrix $E = A - KC$ has the stable eigenvalues $\{-8, -8\}$.

A simple calculation using MATLAB gives $K = \begin{bmatrix} 12.00 \\ -30.67 \end{bmatrix}$.

By Theorem 3, a local exponential observer for the Lotka-Volterra population biology system (28)-(29) around the unique positive equilibrium point $(x_1^*, x_2^*) = (1.5, 1)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(4 - 2z_1 - z_2) \\ z_2(-2 + 2z_1 - z_2) \end{bmatrix} + \begin{bmatrix} 12.00 \\ -30.67 \end{bmatrix} [y - z_1] \tag{35}$$

For simulations, we choose the initial conditions as $x(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $z(0) = \begin{bmatrix} 8 \\ 0.5 \end{bmatrix}$.

Figures 2-3 depict the exponential convergence of the observer states z_1 and z_2 of the system (35) to the states x_1 and x_2 of the Lotka-Volterra population biology system (28)-(29).

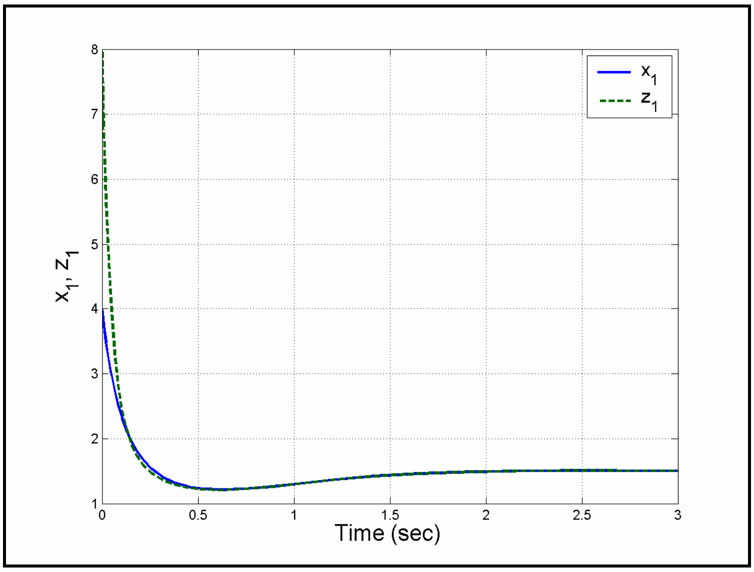

Figure 2. Synchronization of the States x_1 and z_1

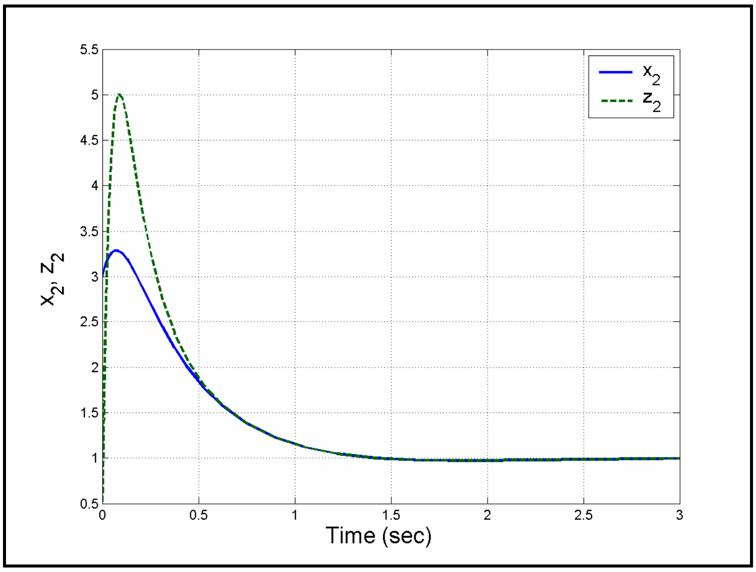

Figure 3. Synchronization of the States x_2 and z_2

6. Conclusions

In this paper, we described a Lotka-Volterra population biology model with negative feedback. We showed that for this biological model, the predator and prey species have stable coexistence. Then we achieved ecological monitoring of the population biology model by constructing a nonlinear exponential observer for the population biology model under study. The nonlinear observer design for the population biology model was constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the Lotka-Volterra population biology model and the prey population size as the output function. Numerical example and MATLAB simulations were shown to illustrate the ecological monitoring or the nonlinear observer design for the two-species Lotka-Volterra population biology model with negative feedback.

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