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Some Biomathematical Models Applying the Adomian Method

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Abstract : The approximate interpretation of some natural phenomena has led to introduce in certain types of differential equations changes in the temporal variable called delays, which makes these equations and their solutions have a more consistent behavior with reality. These equations, called differential equations with delay require complex methods for their solution and in most cases, only a numerical approximation is achieved. In this article we initially show a theoretical development on the decomposition method applied to ordinary differential equations with delay in which the most important properties were studied. Subsequently, the most relevant Adomian polynomials were tested; some biological models that involve differential equations with delay and integro-differential equations were solved. Finally, the numerical comparison was made with other approximation methods and the convergence of the method in some solutions was analysed.
Keywords: Biomathematical Models, Adomian Method.

Introduction

The Adomian Decomposition Method is of great importance in the resolution of non-linear differential equations with initial value and at the border, with the need to know methods that are generally of the semi-analytical numerical type using solutions[1,2,3] of the form $u(x)=\sum_{n=0}^{\infty}u_n$. In general, this method consists of converting a differential equation of real domain into a simpler one (iterative equation) of natural domain. We use the definition and a series of theorems that are applied depending on the characteristic of each of the terms of the differential equation to transform. In essence, this method has its reason for being in the so-called Adomian polynomials and their serial solution, since it is sought to reach an infinite series that represents the approximation of the solution of the equation[4,5].

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The Adomian decomposition method basically consists of a process of semi-analytic numerical type which is applied to all the members of the equation under study, iterating with the values of the variable k in order to obtain the terms $u_0, u_1, u_2, \ldots$ necessary to obtain an approximate solution $u(x)$ always bearing in mind the existence of a reverse differential operator.

The theory and application[6] of differential equations with delay is a very important topic within the dynamic systems, physics and applied mathematics, as well as a particular case of functional differential equations. It deals with models in which the functions at each instant depend not only on a time t, but also on the values for previous instants, which means that the unknown function and its derivatives are evaluated in different arguments. Some important applications are presented in models of population dynamics, heat transfer, biological controls among others. As an interesting example, in the logistic equation it is assumed that the birth rate of a certain organism depends instantaneously on changes in population size, however there are multiple phenomena that affect the process generating a delay. All events of nature require the existence of the time variable to occur or complete a cycle. Some of them take so little time that they do not generate perceptible drawbacks, while others such as those that compromise biological[7,8] processes are highly affected in data collection and in scientific measurement. This subtle observation detected long ago was the reason why in 1948 Hutchinson modified his equation introducing a delay in the growth rate.

In general, the main advantage of the method is the fact that it provides an analytical approach, in many cases an exact solution in a rapidly convergent series. The current results both in the theoretical and computational part show that differential equations with delay are a revealing model of the complex dynamics present in nature[9,10].

Experimental

The theory of the method

Given the equation $Fu = h(t)$, with F is a nonlinear operator[11,12] with linear and nonlinear terms, we can write it in the form $Lu + Ru + Nu = h$. Here, L is the operator for the highest ordered derivative, R is the remainder of the linear term and N is the nonlinear term. Now, applying the inverse operator L^{-1} we obtain

$$L^{-1}Lu = L^{-1}h - L^{-1}Ru - L^{-1}Nu,$$

where L^{-1} is the n-fold definite integration operator from 0 to t.

Therefore, the Adomian method show that the solution $u(t)$ can be written as a series

$$u(t) = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n \quad (1)$$

The term $N(u)$ is defined by Adomian polynomials $N(u) = \sum_{n=0}^{\infty} A_n$, where A_n are the Adomian Polynomial.

The first Adomian polynomials are:

$$A_0 = f(u_0), \quad A_1 = u_1 f^{(1)}(u_0), \quad A_2 = u_2 f^{(1)}(u_0) + \frac{1}{2!}u_1^2 f^{(2)}(u_0)$$

It is important to note that the A_i depend only on the components u_0 to u_i.

Adomian method for differential equations with delay

Consider initially the differential equation[13,14,15] with delay of order n

$$y^{(n)}(x) = f(x, y(x), y(\alpha_1(x)), \ldots, y(\alpha_m(x))), \quad (2)$$

with $x \in [0, b]$ subject to initial conditions

$$\left(y(x), y'(x), \dots, y^{(n-1)}(x)\right) = (h_1(x), h_2(x), \dots, h_n(x)),$$

where $x \in [a, 0]$. Here, a is the minimum of the values $\alpha_i(x) \leq x$ (delay functions) for all $x \in [0, b]$ and $i \in [1, m]$. Here, we assume that $\alpha_i(x)$ and $g_i(x)$ area sufficiently smooth.

Then, we have that the Adomian polynomials are

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda_n}\left[f\left(\sum_{i=0}^{\infty} \lambda^i y_i\right)\right]$$

The calculation of the Adomian polynomials is done as follows. For A_0 we have

$$A_0 = f(y_0), \quad A_1 = f'(y_0)y_1, \ A_2 = \frac{1}{2}f''(y_0)y_1^2 + f'(y_0)y_2$$

Results

Model for growth of tumors in mice

The growth of *ascites Ehrlich* type[16,17] tumors in mice is modeled by the following initial value problem with delay

$$\begin{cases} y'(t) = ry(t-\alpha)\left(1 - \frac{y(t-\alpha)}{C}\right) \\ y(t) = y_0(t) \geq 0, \end{cases} \tag{3}$$

for all $t \in [-\alpha, 0]$. Here, $y(t)$ is related to the number of cells (concentration) in the mouse; *r* is the proportionality (net) rate of reproduction of tumor cells; *C* is the storage capacity and α is the delay showing the duration of a cycle of the multiplicity of cells.

Particular case

Consider next the following particular case of the model (3)

$$\begin{cases} y'(t) = 0.1y(0.5t)(1 - y(0.5t)) \\ y(0) = 0.1 \end{cases}$$

We can see that the differential equation associated with this problem can be written in such a way that the Adomian method for equations with delay is easy to apply, that is,

$$y'(t) = 0.1y(0.5t) - 0.1y(0.5t)y(0.5t)$$

Now, rewriting in terms of the operator we have

$$Ly = 0.1y(0.5t) - 0.1y(0.5t)y(0.5t) \tag{4}$$

Therefore, from the initial condition $y(0) = 0.1$ and assuming the existence of the inverse operator L^{-1} we have:

$$L^{-1}(Ly) = L^{-1}(0.1y(0.5t) - 0.1y(0.5t)y(0.5t)) \tag{5}$$

from where

$$y(t) = 0.1 + L^{-1}(0.1y(0.5t) - 0.1y(0.5t)y(0.5t)) \tag{6}$$

In general, we can see that the recursive formula and approximate solution are obtained as follows

$$y_{k+1}(t) = L^{-1}\big(0.1y_k(0.5t) - 0.1y_k(0.5t)y_k(0.5t)\big) \tag{7}$$

Initially we have that $y_0 = 0.1$, then iterating for all k ≥ 0 we have:

If $k = 0$, then $y_1 = 0.009t$; if $k = 1$, then $y_2 = 0.00018t^2$; if $k = 2$, then $y_3 = -0.00333416t^3$; if $k = 3$, then $y_4 = -0.00000928326t^4$ and so on. Therefore, we obtain

$$\begin{gathered} y(t) = y_0t^0 + y_1t^1 + y_2t^2 + \cdots \\ = 0.1 + 0.009t + 0.00018t^2 - 0.00333416t^3 - 0.00000928326t^4 + \cdots \end{gathered}$$

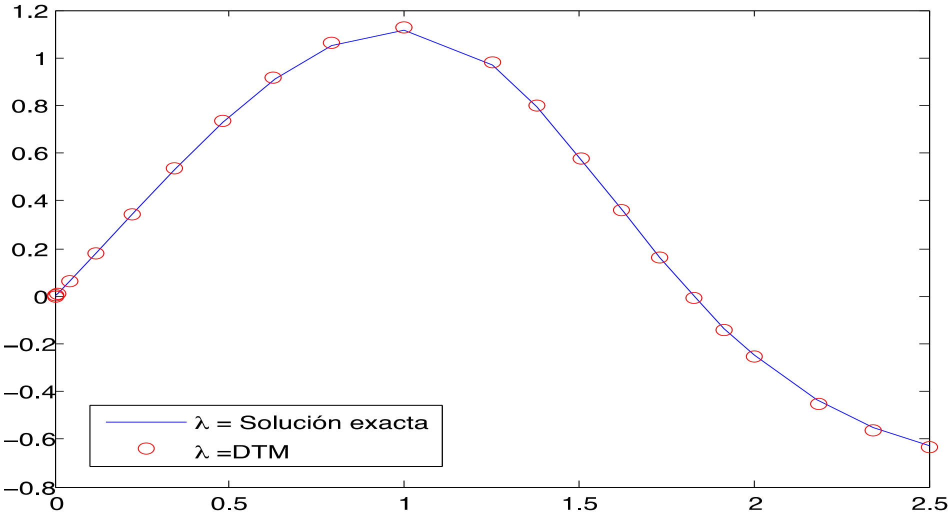

Figure 1. Model for tumor growth.

Population dynamics

Next we present the model[18] in which the population of a certain species given by the function $y(x)$ where its growth rate (increasing or decreasing) is denoted by α. Thus, if the population increases, we will then have a shortage of food as well as space in the habitat.

In general, the behavior of this type of phenomena is modeled by the following initial value problem

$$\begin{cases} y'(x) = \alpha\big(\beta - y(x)\big)y(x), \ x \geq 0 \\ y(x_0) = y_0 \end{cases} \tag{8}$$

Prediction model

Assuming that the growth rate is dependent on the population in previous generations, we obtain the model with delay[19]

$$y'(x) = \alpha(\beta - y(x-\tau))y(x) \tag{9}$$

Now, making the change of variable $z(x) = \alpha\tau\, y(\tau x)$ we have

$$z'(x) = y'(\tau x)(\alpha\tau^2)$$

Therefore, from $y(\tau x)$ we can see that

$$y'(\tau x) = \alpha\left(\beta - y\big(\tau(x-1)\big)\right)y(\tau x),$$

or

$$\begin{aligned} z'(x) &= \left(\alpha\tau\beta - \alpha\tau y\big(\tau(x-1)\big)\right)\alpha\tau y(\tau x) \\ &= (\alpha\tau\beta - z(x-1))z(x) \end{aligned}$$

Finally, returning the change of variable of z by y and $\alpha\tau\beta$ by β we obtain

$$y'(x) = (\beta - y(x-1))y(x) \tag{10}$$

Particular case 1.

As particular case of (8) we have the following model

$$\begin{cases} y'(t) = 3.5y(t)\left(1 - \frac{y(t-0.74)}{19}\right) \\ y(0) = 19.001 \end{cases} \tag{11}$$

Then, to apply the Adomian method we rewrite the differential equation with retardation in the form

$$y'(t) = 3.5y(t) - 0.184211y(t)y(t-0.74)$$

Now, rewriting in terms of the operator we have

$$Ly = 3.5y(t) - 0.184211y(t)y(t-0.74)$$

Therefore, of the initial condition $y(0) = 19.001$ and assuming the existence of the inverse operator L^{-1} we have

$$L^{-1}(Ly) = L^{-1}(3.5y(t) - 0.184211y(t)y(t-0.74)) \tag{12}$$

from where,

$$y(t) = 19.001 + L^{-1}(3.5y(t) - 0.184211y(t)y(t-0.74))$$

In general, we can see that the recursive formula and approximate solution are obtained

as follows:

$$y_{k+1} = L^{-1}(3.5y_k(t) - 0.184211y_k(t)y_k(t - 0.74))$$

Initially we have that $y_0 = 19.001$, then iterating for all $k \geq 0$ we obtain an approximation to the real solution worked by other types of techniques, as evidenced by the attached figure.

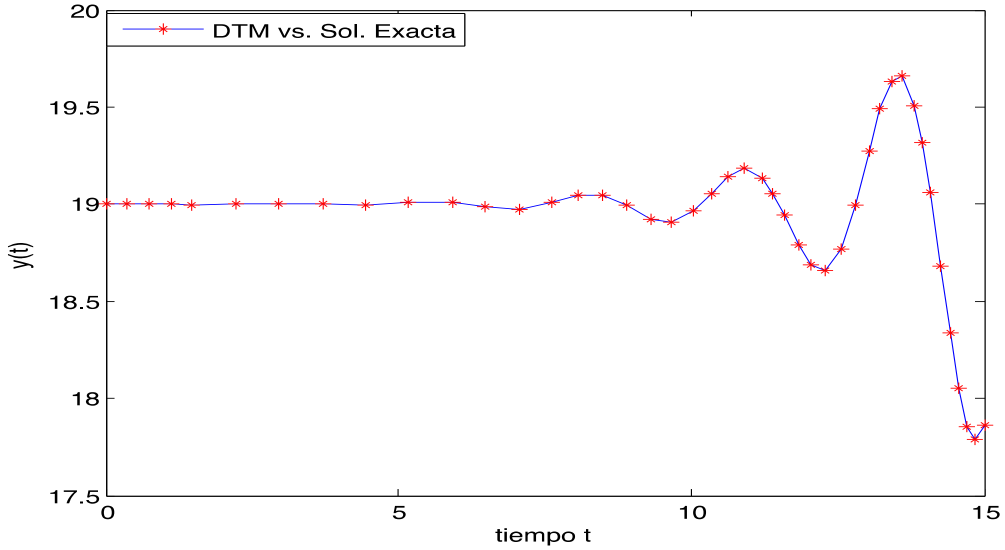

Figure 2. Population dynamics.

Particular case 2.

Now let's consider another particular model[20] of the form (8), that is,

$$\begin{cases} y'(t) = -3y(t-1)(1 + y(t)) \\ y(0) = 0.1 \end{cases} \tag{9}$$

As in the previous case, we can write this equation in such a way that the Adomian decomposition method is applicable without any problem, that is, the problem (9) assumes the form

$$y'(t) = -3y(t-1) - 3y(t)y(t-1)$$

Therefore, of the initial condition $y(0) = 0.1$ and assuming the existence of the inverse operator L^{-1} we have

$$L^{-1}(Ly) = L^{-1}(-3y(t-1) - 3y(t)y(t-1)),$$

from where

$$y(t) = 0.1 + L^{-1}(-3y(t-1) - 3y(t)y(t-1))$$

In general, we can see that the recursive formula and approximate solution are obtained as follows:

$$y_{k+1}(t) = L^{-1}(-3y_k(t-1) - 3y_k(t)y_k(t-1)) \tag{10}$$

After iterating the recurrence equation (10), we obtain an excellent approximation with respect to the original evaluated with computational technique (see Figure 3).

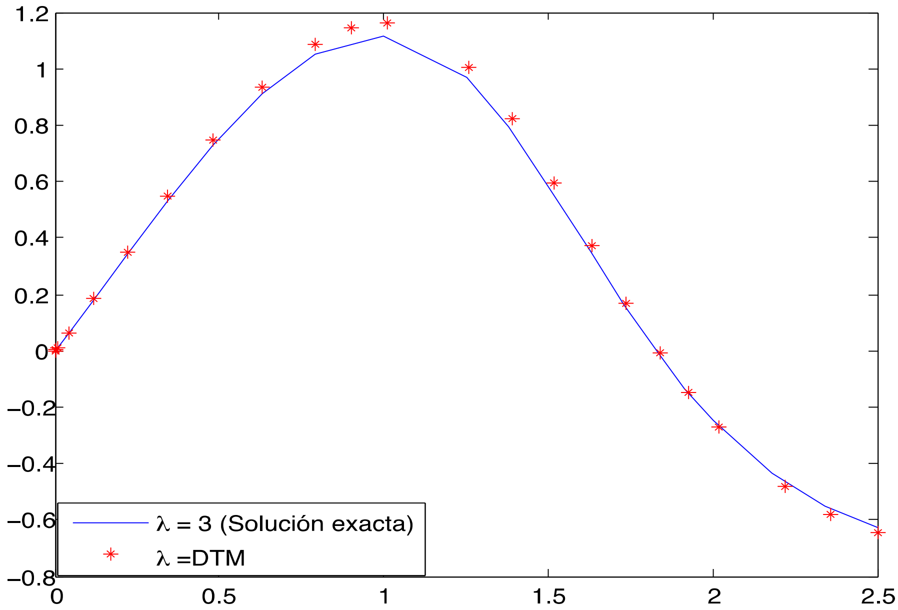

Figure 3. Population dynamics.

Conclusion

In this paper we show the most relevant properties of the Adomian decomposition method using its definition. We present the proofs of each of the theorems as well as the properties of the for ordinary differential equations without delay, as well as the properties from the definition of the inverse operator and its linearity.

In the application of the Adomian method to differential equations with delay, we initially used elementary examples that allowed illustrating and comparing its solution with the exact solution found by analytical methods, thus visualizing the behavior regarding the convergence of the solutions. We apply Adomian later for problems with a higher degree of complexity, such as the value problems of the integer-differential type with different types of delay. We generalize the method to be able to apply it to more complex models (models of Biological type) of differential equations with delay, showing again the fast convergence in comparison with its exact solution.

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References

1. Arboleda, A., Resolución de Ecuaciones Diferenciales No Lineales por el Método de Transformación Diferencial, Tesis de Maestría en Enseñanza de la Matemática, Universidad Tecnológica de Pereira, Colombia, 2012.
2. Cárdenas A, P.P., The Zhou's method for solving the white-dwarfs equation. Applied Mathematics, 2013, 10C(4), 28-32.
3. Cárdenas A, P.P., An Iterative Method for Solving Two Special Cases of Lane-Emden Type Equation. American Journal of Computational Mathematics, 2014, 5(4), 243-253.
4. Abello, C., Resolución de Modelos Físico-Matemáticos No Lineales por el Método de Descomposición

de Adomian, Tesis de Maestría en Enseñanza de la Matemática, Universidad Tecnológica de Pereira, Colombia, 2016.

5. Bani, M.; Amundsen, D., Oscillatory traveling waves for a population diffusion model with two age and nonlocality induced by maturation delay. Applied Mathematics, 2014, 309-324.
6. Abazari, R.; Kilicman, A., Application of Differential Transform Method on Non- linear Integro-Differential Equations with Proportional Delay. Neural Computing and Applications, 2014, 24, 391-397.
7. Biazar, J.; Islam, M.; dIslam, M., Differential Transform Method for Special Systems of Integral Equations. Journal of King Saud University (Science), 2012, 24, 211-214.
8. Liu, B., Zhou, X.; Du, Q., Differential Transform Method for Some Delay Differential Equations. Applied Mathematics, 2015, 6, 585-593.
9. Raslan, K.; Zain, F., Comparison study between differential transform method and Adomian decomposition method for some delay differential equations. Int. J. Phys. Sci., 2013, 17, 744- 2013.
10. Fadhel, F.S.; Mohammed, J., Extend differential transform methods for solving diferential equations with multiple delay., Applied Mathematics, 2011, 24(03), 10.
11. Mirzaee, F.; Latifi, L., Numerical Solution of Delay Differential Equations by Diffe- rential Transform Method., Mathematics Department, Malayer University, Malayer, Iran. 2012, 20(78).
12. Liz, E., Sobre ecuaciones diferenciales con retraso, dinámica de poblaciones y números primos., Materials Matematics, 2006, p. 24.
13. Shampine, L.; Thompson, S., Solving Delay Differential Equations with dde23. Mathematics Department, Southern Methodist University. Dallas, 2000, TX 75275.
14. Kumar, A.; Arora, R. Solutions of the coupled system of Burgers' equations and coupled Klein-Gordon equation by RDT Method., International Journal of Advances in Applied Mathematics and Mechanics, 2001, 1(2), 103-115.
15. Ayaz, F. Solutions of the system of differential equations by differential transform method., Applied Mathematics and Computation, 2004, 147, 547-567.
16. Blanco, L.; Estrella E. Solving delay differential systems with history functions by the Adomian decomposition method., Appl. Math. Comput, 2013, 218, 5994-6011.
17. Smarda, Z.; Diblik, J. Extension of the differential transformation method to non- linear differential equations and integro-differential equations with proportional delays., Advances in Difference Equations, 2013, 69, 1-12.
18. Valli, D.; Kurths, J. Synchronization in coupled Ikeda delay systems., The European Physical Journal, 2014, 223,1465-1479.
19. Karako, F.; Bereketoglu, H. Solutions of Delay Differential Equations by Using Differential Transform Method., International Journal of Computer Mathematics. 2009, 86, 914- 923.
20. Candan, T.; Dahiya, R. Oscillation behavior of n-th order neutral differential equations with continuous delay., Journal of Mathematical Analysis and Applications. 2012, 29, 105-112.

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