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Left Weakly Prime, Semiprime Ideals In Ternary Semigroups

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Abstract : Weakly prime and weakly semiprime ideals in ordered semigroups have been introduced and studied by N. Kehayopulu in [4]. In this paper we introduced and studied the left weakly prime and weakly semiprime ideals of ternary semigroups. We also prove that any left ideal of ternary semigroup is the intersection of all irreducible left weakly prime ideals of ternary semigroup containing it.

Key Words: Ternary semigroup, left weakly prime, left weakly semiprime, left strongly prime, irreducible, strongly irreducible, left semi regular.

1.Introduction :

Weakly prime and weakly semiprime ideals in ordered semigroups have been introduced and studied by N. Kehayopulu in [4] who gave the characterizations of Weakly prime and weakly semiprime ideals. In this paper, we introduced and studied the Weakly prime and weakly semiprime ideals of ternary semigroups.

MSC 2010 Subject classification: 20M12, 20M17.

2. Main Result:

Definition 2.1: Let T be a ternary semigroup and $S \subseteq T$. S is called left (lateral, right) weakly prime if for any left(lateral, right) ideals A,B,C of T such that $ABC \subseteq S$ we have $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$.

Definition 2.2: Let T be a ternary semigroup and $S \subseteq T$. S is called left (lateral, right) strongly prime if for any left(lateral, right) ideals A,B,C of T such that $ABC \cap BCA \cap CAB \subseteq S$ we have $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$.

Definition 2.3: Let T be a ternary semigroup and $S \subseteq T$. S is called left (lateral, right) weakly semiprime if for any left(lateral, right) ideals A of T such that $\{0\} \neq A^3 \subseteq S$ we have $A \subseteq S$.

S is called left (lateral, right) weakly semiprime ideal of T if S left (lateral, right) ideal which is left (lateral, right) weakly semiprime.

Definition 2.4: A ternary semigroup is called a fully left (lateral, right) weakly prime ternary semigroup if all its left (lateral, right) ideals are left (lateral, right)) weakly prime left (lateral, right) ideals.

A ternary semigroup is called a fully left (lateral, right) weakly semi prime ternary semigroup if all its left (lateral, right) ideals are left (lateral, right)) weakly semi prime left (lateral, right) ideals.

One can easily see that the concepts of the left weakly prime, left strongly prime and the left weakly semiprime are the extension of the concepts of weakly prime and weakly semiprime of ternary semigroup.

The left strongly prime ideals is left weakly prime ideals and the left weakly prime ideals are left weakly semiprime ideals but the converse is not true. We show it by the following example.

Example 2.5: We consider the set $T=\{p,q,r,s,t\}$, defined by multiplication and the order below

	p	q	r	s	t
p	p	t	r	s	t
q	p	q	r	s	t
r	p	t	r	s	t
s	p	t	r	s	t
t	p	t	r	s	t

Clearly T is a ternary semigroup.

The right ideals of T are the sets $\{p,r,s,t\}$ and *T*.

The left ideals of T are the sets $\{p\}$, $\{r\}$, $\{p,r\}$, $\{p,s\}$, $\{r,t\}$, $\{p,r,s\}$, $\{p,r,t\},\{q,r,t\},\{p,q,r,t\},\{p,r,s,t\}$ and *T*.

The ideals of T are the sets $\{p,r,s,t\}$ and *T*.

The left weakly prime ideals of T are $\{p\}$, $\{r\}$, $\{p,r\}$, $\{p,s\}$, $\{p,r,s\}$, $\{q,r,t\}$, $\{p,q,r,t\}$, $\{p,r,s,t\}$ and *T*.

The left ideals $\{r,t\}$ and $\{p,r,t\}$ are not left weakly prime ideals. In fact,

$\{p,r\}\{p,r,t\}\{q,r,t\}=\{r,t\}$ but $\{p,r\}\not\subseteq\{r,t\}$ or $\{p,r,t\}\not\subseteq\{r,t\}$ or $\{q,r,t\}\not\subseteq\{r,t,\}$.

$\{p,r,s\}\{p,r,s,t\}\{q,r,t\}=\{r,t\}\subseteq\{p,r,t\}$ but $\{p,r,s\}\not\subseteq\{p,r,t\}$ or $\{p,r,s,t\}\not\subseteq\{p,r,t\}$ or $\{q,r,t\}\not\subseteq\{p,r,t\}$.

The left strongly prime ideals of T are $\{p,s\}$, $\{q,r,t\},\{p,q,r,t\}$ and $\{p,r,s,t\}$

All left ideals of *T*are left weakly semiprime,so *T*is a fully left weakly semiprime ternary semigroup. However , left ideals {r,t}and {p,r,t} are left weakly semiprime ideals but they are not left weakly prime ideals and left strongly prime ideals.

Infact, $\{p,q,r,t\}\{q,r,t\}\{p,r,s,t\}\cap\{q,r,t\}\{p,r,s,t\}\{p,q,r,t\}\cap\{p,r,s,t\}\{p,q,r,t\}\{q,r,t\}=\{r,t\}$,

But $\{p,q,r,t\}\not\subseteq\{r,t\}$, $\{q,r,t\}\not\subseteq\{r,t\}$ and $\{p,r,s,t\}\not\subseteq\{r,t\}$

$\{p,q,r,t\}\{q,r,t\}\{p,r,s,t\}\cap\{q,r,t\}\{p,r,s,t\}\{p,q,r,t\}\cap\{p,r,s,t\}\{p,q,r,t\}\{q,r,t\}=\{r,t\}$
$\subseteq\{p,r,t\}$,

But $\{p,r,s,t\} \not\subseteq \{p,r,t\}$, $\{q,r,t\} \not\subseteq \{p,r,t\}$ and $\{p,q,r,t\} \not\subseteq \{p,r,t\}$.

We now introduce an irreducible left (lateral, right) ideal, strongly irreducible left (lateral, right) ideal and characterize irreducible, strongly irreducible left (lateral, right) ideals.

Definition2.6: A left (lateral, right) ideal L of a ternary semigroup T is called an irreducible left(lateral, right) ideal if $L_1 \cap L_2 \cap L_3 = L$ implies either $L_1 = L$ or $L_2 = L$ or $L_3 = L$ for every left ideal L_1, L_2 and L_3 of T.

Definition2.7: A left(lateral, right) ideal L of a ternary semigroup T is called an strongly irreducible left(lateral, right) ideal if $L_1 \cap L_2 \cap L_3 \subseteq L$ implies either $L_1 \subseteq L$ or $L_2 \subseteq L$ or $L_3 \subseteq L$ for every left ideal L_1, L_2 and L_3 of T.

Every strongly irreducible left ideal of a ternary semigroup is an irreducible left ideal but the converse is not true.

The following exampleshows that an irreducible left ideal of a ternary semigroup may not be a strongly irreducible left ideal.

Example 2.8: We consider the set $T = \{p,q,r,s,t,u\}$ defined by multiplication and the order below.

	p	q	r	s	t	u
p	s	t	s	s	t	s
q	s	t	s	s	t	s
r	s	t	s	s	t	s
s	s	t	s	s	t	s
t	s	t	s	s	t	s
u	p	q	r	s	t	u

For an easy way we can check that T is a ternary semigroup.

All left ideals of T are the sets:

$\{t\},\{q,t\},\{s,t\},\{p,s,t\},\{q,s,t\},\{r,s,t\}\{s,t,u\},\{p,r,s,t\},\{p,q,s,t\}\ \{p,s,t,u\}$

$\{q,r,s,t\}\{q.s.t,u\}\{r,s,t,u\},\{p,q,r,s,t\},\{p,q,s,t,u\},\{p,r,s,t,u\},\{q,r,s,t,u\}$

and*T*.

All irreducible left ideals are the sets:

$\{q,t\},\{p,r,s,t\},\{p,q,s,t\},\{p,s,t,u\},\{q,r,s,t\}\{q,s,t,u\},\{r,s,t,u\},\{p,q,r,s,t\}$

$\{p,q,s,t,u\}\{p,r,s,t,u\}\{q,r,s.t,u\}$ and*T*.

All strongly irreducible left ideals are the sets:

$\{q,t\},\{p,q,r,s,t\}\{p,q,s,t,u\}\{p,r,s.t,u\}\{q,r,s,t,u\}$ and*T*.

Theorem 2.9 : Every strongly irreducible left weakly semiprime ideal of a ternary semigroup T is left strongly prime ideal and the concepts of left strongly prime ideals and left weakly prime ideals of T are the same.

Proof: Let S be a left weakly semiprime ideal of a ternary semigroup T. Let A,B,C be any three left ideals of T such that $ABC \cap BCA \cap CAB \subseteq S$. By known theorem , we have

$(A \cap B \cap C)^3 \subseteq ABC, (A \cap B \cap C)^3 \subseteq BCA$ and $(A \cap B \cap C)^3 \subseteq CAB.$

so $(A \cap B \cap C)^3 \subseteq ABC \cap BCA \cap CAB \subseteq S$.

Since S be a left weakly semiprime ideal,then $A \cap B \cap C \subseteq S$.

As S is a strongly irreducible left ideal, hence either $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$.

Thus S is a left strongly prime ideal.

Obviously, left strongly prime ideals are left weakly semiprime ideals.

Conversely, let S be a left weakly semiprime ideal of a ternary semigroup T, A,B,C be any three left ideals of T such that $ABC \cap BCA \cap CAB \subseteq S$.

Since $ABC \cap BCA \cap CAB$ is a left ideal of T and

$ABC \cap BCA \cap CAB \subseteq ABC \cap BCA \cap CAB$.

By hypothesis either $ABC \subseteq ABC \cap BCA \cap CAB$ or

$BCA \subseteq ABC \cap BCA \cap CAB$ or $CAB \subseteq ABC \cap BCA \cap CAB$,

So either $ABC \subseteq S$ or $BCA \subseteq S$ or $CAB \subseteq S$.

Since S is a left weakly prime ideal of T , then either $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$.

That is S is a left strongly prime ideal of T.

Theorem 2.10 : Let T be a ternary semigroup. Then the following conditions are equivalent:

i) The left ideals of T form a chain under inclusion.
ii) Every left ideal of T is strongly irreducible.
iii) Every left ideal of T is irreducible

Proof: (i) $\Rightarrow$ (ii)

Let S be a left ideal of T and let A,B,C are three left ideals of T such that $A \cap B \cap C \subseteq S$.

Since the left ideals of T form a chain under inclusion, so we have $A \subseteq B,C$ or $B \subseteq A,C$ or $C \subseteq A,B$.

Thus either $A \cap B \cap C = A$ or $A \cap B \cap C = B$ or $A \cap B \cap C = C$.

Hence $A \cap B \cap C \subseteq S$ .implies either $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$. i.e, T is strongly irreducible.

(iii) $\Rightarrow$ (iii)

Let S be a left ideal of T and let A,B,C are three left ideals of T such that $A \cap B \cap C = S$.

Then $S \subseteq A, S \subseteq B$ and $S \subseteq C$ .By hypothesis either $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$,

so either $A = S$ or $B = S$ or $C = S$ .i.e, S is irreducible.

(iii) $\Rightarrow$ (i)

Let A,B,C are three left ideals of T, $A \cap B \cap C$ is a left ideal of T.

So, by hypothesis either $A = A \cap B \cap C \; or \; B = A \cap B \cap C \; or \; C = A \cap B \cap C$.

It implies either $A \subseteq B,C$ or $B \subseteq A,C$ or $C \subseteq A,B$.

That is, the left ideals Tform a chain under inclusion.

Definition 2.11: An element a of a ternary semigroup T is a left(lateral, right) semi regular element if $a = pqarsatua$ (*a=paqrastau, a=apqarsatu)* for some $p,q,r,s,t,u \in T$.

Definition 2.12: T is called left(lateral, right) semiregular if all elements of T are left(lateral, right) semiregular.

T is called quasi-completely regular if T is a left or right semiregular.

Equivalently definition $a \in TTaTTaTTa$ for every $a \in T$.

Example 2.13: We consider the set $T = \{p, q, r, s, t\}$, defined by multiplication and the order below

	p	q	r	s	t
p	p	t	r	s	t
q	p	q	r	s	t
r	p	t	r	s	t
s	p	t	r	s	t
t	p	t	r	s	t

Clearly T is a ternary semigroup.

for any $x \in T$. If $x \neq q$, then $x \in xTTxTTxTT = \{p, r, s, t\}$,

$TTxTTxTTx = \{x\}$

so $x \in xTTxTTxTT; xTTxTTxTT$.

$qTTqTTqTT = T^3 = T$ and $TTq = \{p, q, r, s, t\}$,

so $q \in qTTqTTqTT$ and $q \in TTqTTqTTq$.

It implies that T is left semiregular and right semiregular.

Remark 2.14 : Every left weakly semiprime ideal of a ternary semigroup T is left semiregular.

Theorem 2.15 : Every left ideal of a ternary semigroup T is left strongly prime if and only if T is left semi regular and the left ideals of T form a chain under inclusion.

Proof : Let any left ideal of T be left strongly prime, then every left ideal of T is left weakly semiprime. So T is left semiregular.(By remark 2.14).

Now we prove that the left ideal of T form a chain under inclusion.

Let A,B,C are any three ideals of T, then $ABC \cap BCA \cap CAB \subseteq A \cap B \cap C$.

Since every left ideal of T is left strongly prime and $A \cap B \cap C$ is a left ideal of T.

So $A \cap B \cap C$ is left strongly prime.

Hence either $A \subseteq A \cap B \cap C$ or $B \subseteq A \cap B \cap C$ or $C \subseteq A \cap B \cap C$.

If $A \subseteq A \cap B \cap C$ then $B \subseteq A$ and $C \subseteq A$.

If $B \subseteq A \cap B \cap C$ then $A \subseteq B$ and $C \subseteq B$.

If $C \subseteq A \cap B \cap C$ then $A \subseteq C$ and $B \subseteq C$.

Conversely, let S be an arbitrary left ideal of T and let A,B,C be three left ideals of T such that

$ABC \cap BCA \cap CAB \subseteq S$.

Since T is left semiregular, we have

$$A \cap B \cap C = (A \cap B \cap C)^3 \subseteq (ABC \cap BCA \cap CAB) \subseteq S = S .$$

Since the left ideals of T form a chain under inclusion. So we have $A \subseteq B$ or $B \subseteq C$ or $C \subseteq A$.

Then either $A \cap B \cap C = A$ or $B \cap C \cap A = B$ or $C \cap A \cap B = C$.

Hence $A \cap B \cap C \subseteq S$ .implies either $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$. i.e, *S* is stronglyprime.

Theorem 2.16: Let *S* be a left weakly prime ideal of a ternary semigroup T with 0, defined by 0.0.*a*= o.*a*.0 = *a*.0.0 , then (*S*,*a*) = { x ∈ T/ *xaa*∈ S} is a left weakly prime ideal of *T*, for any *a* ∈ *T**S*.

Proof :since *S* be a left weakly prime ideal of a ternary semigroup T, then obviously *0*∈*T*, and hence *0* ∈(*S*,*a*). Therefore (*S*,*a*) ≠ ∅. Let *x*∈(*S*,*a*); b,c ∈*T*. We have *(bcx)aa=b(cxa)a=bc(xaa)* ∈*S*.

Infact *xaa*∈*S*and *S* is a left ideal of *T*. Thus bcx ∈(*S*,*a*). Therefore (*S*,*a*) is a left ideal of *T*.

Let *A*,*B*,*C* be three left ideals of *T* such that $ABC \subseteq (S,a)$ because *Aaa*,*Baa*,*Caa* are left ideal of *T*.

$$\Rightarrow Aaa \subseteq S \text{ or } Baa \subseteq S \text{ or } Caa \subseteq S. \Rightarrow A \subseteq (s,a) \text{ or } B \subseteq (s,a) \text{ or } C \subseteq (s,a).$$

Theorem 2.17: Let *T* be a ternary semigroup with identity element *e* as *e.e.a* = *e.a.e* = *a.e.e* = *a.S* is a left weakly prime ideal of *T* , if *J* = *{a*∈ *T/ aTT* $\subseteq$ *S}* is a non empty subset of *T* , then *J* is the maximal ideal of *T* contained in *S* and *J* is left weakly prime.

Proof : First we prove *J* = *{a*∈ *T/ aTT* $\subseteq$ *S}* is an ideal of *T*. Let $a \in J$ and $x, y \in T$, then

$$(xya)TT = x(yaT)T = xy(aTT) \subseteq xyS \subseteq S \Rightarrow xya \in J.$$
$$\text{Now} (axy)TT = a(xyT)T = ax(yTT) = (aTT)xy \subseteq Sxy \subseteq S \Rightarrow axy \in J$$
$$\text{and} (xay)TT = x(ayT)T = xa(yTT) = x(aTT)y \subseteq xSy \subseteq S \Rightarrow xay \in J.$$

Therefore *J* is an ideal of *T*.Clearly $T \subseteq S$. Let *K*be an ideal of *T* such that $K \subseteq S$.

Let $x \in K$. Then $xTT \subseteq K \subseteq T$, so $x \in J$. Thus $K \subseteq J$. Hence*J* is the maximal ideal of *T* contained in *S*. For all left ideals *A*,*B*,*C* of *T* such that $ABC \subseteq J$. Since*S* is a left weakly prime

ideal of *T* and $J \subseteq S$ .So $A \subseteq S$ or $B \subseteq S$ or $C \subseteq S$.

If A⊈ J or B ⊈ J or C ⊈ J, then

$J \cup A, J \cup B$ and $J \cup C$ are left ideal of T, and $J \subset J \cup A \subseteq S$ or $J \subset J \cup B \subseteq S$ or

$J \subset J \cup C \subseteq S$. The maximality of *J* implies that $J \cup A = S$ or $J \cup B = S$ or $J \cup C = S$.

Since $S = J \cup A \subseteq S \cup A \subseteq S$ or $S = J \cup B \subseteq S \cup B \subseteq S$ or $S = J \cup C \subseteq S \cup C \subseteq S$.

Hence $J \cup A = S \cup A$ or $J \cup B = S \cup B$ or $J \cup C = S \cup C$. Then we have *J*=*T* impossible.

Remark2.18: Every left ideal of a ternary semigroup *T* is a left semiregular.

Theorem2.19: Let *T* be a ternary semigroup. If *T* is fully left weakly prime, then T is left semiregular and the ideals of *T* form a chain under set inclusion.

Proof : Let *T* be fully left weakly prime and *L* be any left ideal of *T*. Thus L = L^3 and hence T is a left semiregular. Let *A*,*B*,*C* be three ideals of *T* , then $ABC \subseteq A \cap B \cap C \Rightarrow A \subseteq A \cap B \cap C$ or

$$B \subseteq A \cap B \cap C \text{ or } C \subseteq A \cap B \cap C \Rightarrow A \subseteq B, C \text{ or } B \subseteq C, A \text{ or } C \subseteq A, B.$$

Theorem2.20: If T is a left semiregular ternary semigroup such that the left ideals of T form a chain under inclusion , then every left ideal of T is left weakly prime.

Proof :.Let *A,B,C* be three ideals of *T*, such that $ABC \subseteq T$. Since the left ideals of T form a chain, so without loss of generality, we assume that $A \subseteq B; B \subseteq C; C \subseteq A$. By T is a left semiregular and $ABC \subseteq T$, we have $A = A^3 = AAA \subseteq ABC \subseteq T$. *So* $A \subseteq T$. Hence T is left weakly prime.

Theorem2.21: Let *T* be a ternary semigroup such that the left ideals of T form a chain under set inclusion, then T is fully left weakly prime if and only if T is left semiregular.

Proof : The proof of the theorem follows by theorem 2.19 and 2.20.

Theorem 2.22 : Let *a* be a left semi regular element of a ternary semigroup *T* . If *S* is a left ideal not containing *a,* then there exist an irreducible left weakly prime ideal *P*of *T* containing *S*and not containing *a.*

Proof : Let $\{A_i / i \in \Delta\}$ be a chain of left ideals of *T* containing *S* and not containing *a,*then $\bigcup A_i$ is a left ideal of *T* containing *S*and not containing *a*. Therefore, by Zorn's lemma, the set of all left ideals of *T* containing *S*and not containing *a*has a maximal element *M*. Suppose $M = B \cap C \cap D$, where *B,C* and *D* are left ideals of *T* properly containing *M*. Then by the choice of *M* , $a \in B, a \in C$ *and* $a \in D$ .Thus $a \in B \cap C \cap D$ =*M*, Which is a contradiction. Hence , M is an irreducible left ideal of *T*. Let L_1, L_2 *and* L_3 be three left ideals such that $L_1L_2L_3 \subseteq M$; $L_1 \not\subseteq M$,

$L_2 \not\subseteq M$and $L_3 \not\subseteq M$ .Since $L_1 \cup M, L_2 \cup M$ *and* $L_3 \cup M$ are left ideals of T , by the choice of M, we have $a \in L_1, a \in L_2$ and $a \in L_3$. since*a* be a left semiregular element of T, then $a \in TTaTTaTTa \subseteq TTL_1TTL_2TTL_3 \subseteq L_1L_2L_3 \subseteq M$. Hence $a \in M$, which is a contradiction. Hence M is a left weakly prime ideal of T containing *S* and not containing *a*.

We can get easily the following corollary from theorem 2.22

Corollary2.23: Let *a* be a left semi regular element of a ternary semigroup *T*. If *S* is a left ideal not containing a^n for any odd positive integer *n*, then there exist an irreducible left weakly prime ideal *P* of *T* containing *S* and not containing a^n for any odd positive integer *n*.

Theorem 2.24: Let *T* be a left semiregular ternary semigroup. Then any left ideal *S* of *T* is the intersection of all irreducible left weakly prime ideals of *T* containing *S* .

Proof : Let S be a left ideal of T and $\{A_i / i \in \Delta\}$ be the collection of irreducible left weakly prime ideals of *T* containing *S* , then $S \subset \bigcap A_i$. For the reverse inclusion, let $a \notin S$, then by theorem2.22, then there exists an irreducible left weakly prime ideal P of T containing S but not containing a. Thus $a \notin \bigcap A_i$. Hence, $S = \bigcap A_i$.

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