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Numerical Methods – Engineering Applications

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Abstract : In this paper, we present a few selected applications of Numerical methods to other parts of mathematics and to various other fields in general. Numerical methods are rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include chemistry, electrical engineering, operation research. The wide scope of these and other applications has been well documented.
Keywords: Numerical Methods, Engineering Applications.

1.Introduction

Numerical methods provide a way to solve problems quickly and easily compared to analytic solutions. Whether the goal is integration or solution of complex differential equations, there are many tools available to reduce the solution of what can be sometimes quite difficult analytical math to simple algebra.

2. PDEs Contains Two Main Aspects

1. Analytic methods

2. Numerical approximation.

Both the mathematical analysis of the PDEs and the numerical analysis of methods rely heavily on the strong tools of functional analysis. Numerical approximation of PDEs is a cornerstone of the mathematical modeling since almost all modeled real world problems fail to have analytic solutions or they are not known in the scope of pure mathematics because of their complexity[1]. The history of numerical solution of PDEs is much younger than that of analytic methods, but the development of high speed computers nowadays makes the advent of numerical methods very fast and productive. On the other hand, the numerical approximation of PDEs often demands knowledge of several aspects of the problem, in order to understand and interpret the behaviour of expected solutions, or the algorithmic aspects concerned with the choice of the numerical method and the accuracy that can be achieved[2]. The aim of this study is to discuss some modeling problems and provide the knowledge of Finite Element techniques for the numerical approximation of the model equations. Especially the theory and application of finite element methods is a very nice combination of mathematical theory with aspects of functioning, analysing, and applications.

3. Multiple Scales in the Modelling of Real World Problems

Most of the phenomena in nature are concerned with the behaviour of a big number of individual objects which are always in a close interaction with each other. On the other hand, the important features are

visible on a much coarser macro scale, where a mean behaviour of objects is observable. Let us consider a very short list of fields where such behaviour arises in real life problems and where the mathematical investigation is needed to answer the questions stated by the problem.

3.1 Meteorology

It deals with the interaction of air (oxygen, nitrogen, ozone, etc.) and water molecules, as those exchanges are responsible for the behaviour of macroscopic variables like temperature, pressure humidity, and wind. The main objective of meteorological stations is to develop a system which permits reliable monitoring of climate changes[3]. The monitoring is of high importance for like airports, offshore wind parks, etc.

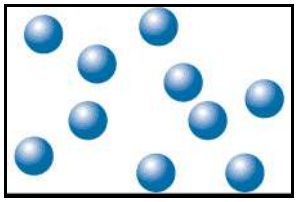


Fig.1 Air molecules in atmosphere

3.2 Civil Engineering

In many aspects of our life a huge amount of different materials are used. Glass, wood, metals, concrete, which are directly use almost every minute in our everyday life. Thus, the modification of materials and prediction of their properties are very important objectives for the manufacturers. In order to produce high quality materials the engineers in industry, among other problems, are very much interested in the elastic behaviour or loading capacity of the material[4]. While it is known that the bonding forces between the atoms of the material are responsible for their physical and chemical properties. So, to manufacture a new product with higher quality, a detailed investigation of the material on the atomic level is not required in most cases. A mathematical model is needed for the quantitative description of the change of material properties under external influences. The concepts of differential equations come to help us as an excellent tool for the development of such a model.

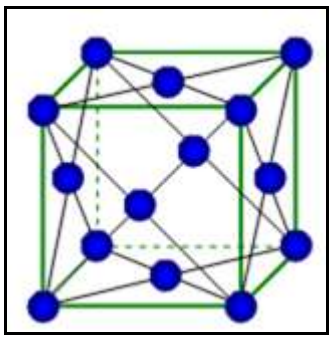


Fig. 2. Crystal Lattice

3.3 Traffic Flow

People spend several hours on their way back home because of the traffic jams on the roads after their hectic work. During the driving process every driver has one's own behaviour which depends on the objectives of being fast and avoiding accidents. So, in this way a driver interacts with other cars. At all times drivers drive unthinkingly in such a manner as to avoid the traffic jams on the roads[5]. Again we need the help of mathematical model which can provide the understanding necessary to make the life of drivers more pleasant. The developed model can serve as ancient model also for other application problems. For example, the traffic jam model is similar to gas flow models which allow for the appearance of shock waves. In aircraft traffic, analogous problems cause noise pollution near airports.

4. Mathematical Modeling

4.1 Density Flux and Conservation

The simplest mathematical models can be developed with the help of density, flux and a conservation law. As examples of a density we can consider some space quantities which can vary in time. Quantities like concentration of a substance or the heat density in a body are two simple examples.

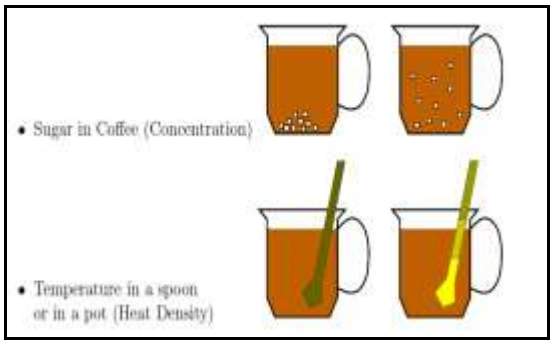
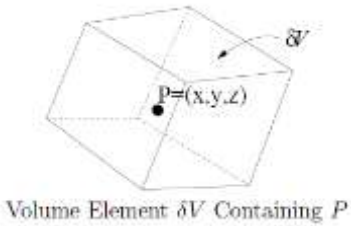


Fig.3. Heat transfer

The mass density is, for sure, the simplest example of density. To define the mass density (density of a material), we consider a point P = (x; y; z) in the space, and let _V be a small volume element containing P[6].



Volume Element δV Containing P

The average mass density _ in _V at time t is equal to the mass contained in _V (which is proportional to the number of molecules), divided by the volume of j_V j:

$$\rho(\delta V, t) = \frac{\text{Mass in } \delta V \text{ at time } t}{|\delta V|}$$

In order to determine the mass density _(P; t) in the point P at time t, we should allow _V to become smaller and smaller.

4.2 FLUX

It is known that single objects like molecules or organisms are in continuous movement. So, we want to define the flow vector (flux) q(P; t) in a point P at time t to be the rate and direction of average movement of the objects. Like in the case of density, the flux can be also defined through a limiting process.

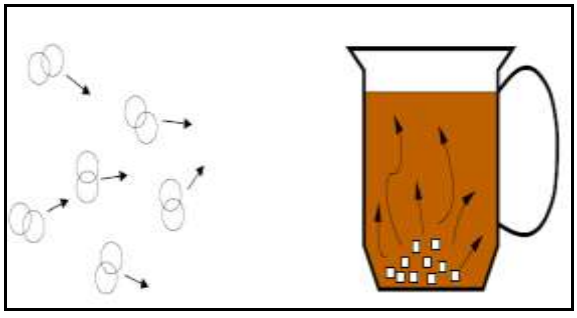


Fig.4. Flux

5.Pdes as a Modeling Tool

5.1 Heat Conduction Equation (1D)

If the left end of the rod is at a higher temperature, then heat energy will be transferred from left to right across the rod toward the colder part. In conduction process motion of the material as a whole is not considered[7]. Thermal energy is conserved in the conduction process. The content of energy depends on density and specific heat cp depends on the temperature T via The heat flux tries to equi-distribute heat over a piece of material. Thus, heat flows from hotter to cooler part. Fourier's law governs the conduction process, in a homogeneous medium the rate of heat flow is directly proportional to the temperature difference along the path of heat flow,

$$e(x,t) = \rho c_p T(x,t)$$

$$q(x,t) = -k\frac{\partial}{\partial x}T(x,t)$$

Where k > 0 is a material parameter and is called the thermal conductivity. The governing equation for one-dimensional heat conduction is given by,

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + f$$

6. Galerkin Method

Galerkin methods are used for converting a continuous operator problem to a distinct problem. It is the method of applying the variation of parameters to a function space, by transforming the equation to a weak formulation.

6.1 Introduction with an Abstract Problem

6.1.1 A Problem in Weak Formulation

Let us introduce Galerkin's method with an abstract problem posed as a weak formulation on a Hilbert space V, namely,

find $u \in V$ such that for all $v \in V, a(u,v) = f(v)$.

Here, $a(\cdot,\cdot)$ is a bilinear form (the exact requirements on $a(\cdot,\cdot)$ will be specified later) and f is a bounded linear functional on V.

6.1.2 Galerkin Dimension Reduction

Choose a subspace $V_n \subset V$ of dimension *n* and solve the projected problem:

Find $u_n \in V_n$ such that for all $v_n \in V_n, a(u_n, v_n) = f(v_n)$.

The equation has not changed and the spaces have changed, converting the problem to a finite-dimensional vector subspace allows us to numerically compute u_n as a finite linear combination of the basis vectors in V_n[8].

The key property of the Galerkin approach is that the error is orthogonal to the chosen subspaces. Since $V_n \subset V$, we can use v_n as a test vector in the equation. We get the Galerkin orthogonality relation for the error, $\epsilon_n = u - u_n$ which is the error between the solution of the original problem, u, and the solution of the Galerkin equation, u_n

$$a(\epsilon_n, v_n) = a(u, v_n) - a(u_n, v_n) = f(v_n) - f(v_n) = 0.$$

6.1.3 Matrix Form

In Galerkin's method the production of a linear system of equations is needed, we build its matrix form, which can be used to compute the solution by a computer program.

Let $e_1, e_2, \ldots, e_n$be a basis for V_n. Then, it is sufficient to use these in turn for testing the Galerkin equation, i.e.: find $u_n \in V_n$such that

$$a(u_n, e_i) = f(e_i) \quad i = 1, \ldots, n.$$

$$u_n = \sum_{j=1}^{n} u_j e_j$$

We expand $u_n$with respect to this basis, and insert it into the equation,

$$a\left(\sum_{j=1}^{n} u_j e_j, e_i\right) = \sum_{j=1}^{n} u_j a(e_j, e_i) = f(e_i) \quad i = 1, \ldots, n.$$

This previous equation is actually a linear system of equations $Au = f$, where

$$A_{ij} = a(e_j, e_i), \quad f_i = f(e_i).$$

6.1.4 Symmetry of The Matrix

The matrix of the Galerkin equation is symmetric if the bilinear form $a(\cdot, \cdot)$is symmetric.

6.1.5 Analysis of Galerkin Methods

As per symmetric bilinear forms,

$$a(u, v) = a(v, u).$$

The application of the standard Galerkin method becomes much easier. Petrov–Galerkin method may be required in the non-symmetric case[9]. The analysis of these methods proceeds in various steps. It is to be showed that the Galerkin equation is a well-posed problem in the sense of Hadamard.

The analysis will mostly rest on two properties of the bilinear form, namely

Boundedness: for all $u, v \in V$ holds

$a(u, v) \leq C\|u\| \, \|v\|$for some constant $C > 0$

Ellipticity: for all $u \in V$ holds

$a(u, u) \geq c\|u\|^2$for some constant $c > 0$.

The norms in the following sections will be norms for which the above inequalities hold (these norms are often called an energy norm).

6.1.6 Well-Posedness of the Galerkin Equation

Since $V_n \subset V$, boundedness and ellipticity of the bilinear form apply to V_n. Therefore, the well-posedness of the Galerkin problem is actually inherited from the well-posedness of the original problem.

7 Quasi-Best Approximation (Céa's Lemma)

The error $u - u_n$between the original and the Galerkin solution admits the estimate.

$$\|u - u_n\| \leq \frac{C}{c} \inf_{v_n \in V_n} \|u - v_n\|.$$

This means, that up to the constant C/c, the Galerkin solution $u_n$is as close to the original solution $u$as any other vector in V_n. In particular, it will be sufficient to study approximation by spaces V_n, completely forgetting about the equation being solved.

Proof

Since the proof is very simple and the basic principle behind all Galerkin methods, by boundedness of the bilinear form (inequalities) and Galerkin orthogonality (equals sign in the middle), we have for arbitrary $v_n \in V_n$:

$$c\|u - u_n\|^2 \leq a(u - u_n, u - u_n) = a(u - u_n, u - v_n) \leq C\|u - u_n\|\|u - v_n\|.$$

Dividing by $c\|u - u_n\|$and taking the infimum over all possible $v_n$yields the lemma.

7.1 Application of Galerkin Method: Example 1

Here, we consider a beam which is simply supported at both ends, as shown in Fig. The overall length of the beam is L[10].

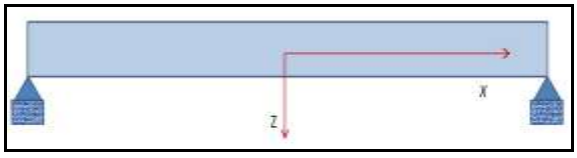


Fig.5. Load distribution

Here z, we assume:

-Material is isotropic.

-Normal uniform load of intensity *q*0 is applied over the length of the beam.

For such a system, the governing equation for normal deflection is:

$EI(d^4w/dx^4) - q_0 = 0$.

The accuracy of Galerkin solution for the above eq. with exact solution.

In such a case, the boundary conditions are:

– Displacement at ends of the beam is zero, i.e. $w = 0$ at $x = \pm L/2$.

– Moment at both beam ends is zero.

At this stage, we select a displacement function, which satisfies the kinematic boundary conditions.

Thus, we assume:

$w^o(x) = A \cos(\pi x/L)$

• Such a displacement function satisfies the displacement BC at both ends of beam. Substituting this function in governing equation gives us error in the force. The relation for this error is:

$E[w(x)] = AEI\cdot(\pi/L)^4\cdot \cos(\pi x/L) - q_0$

• The virtual work done by this error force as defined in the above eq , when integrated over the length of beam is equal to zero. So,

$\int_\Omega E[w(x)]\cdot \varepsilon w1(x)\, dx = 0$, where integration limits are $-L/2$ and $L/2$.

or,

$\int_\Omega [AEI\cdot(\pi/L)^4\cdot \cos(\pi x/L) - q_0]\cdot \varepsilon w_1(x)\, dx = 0.$

• At this stage, we chose w_1 as defined below:

$w_1(x) = A_1 \cos(\pi x/L)$

• Thus, the above eq. can be rewritten as:

$\int_\Omega [AEI\cdot(\pi/L)^4\cdot \cos(\pi x/L) - q_0]\cdot \varepsilon A_1 \cos(\pi x/L)\, dx = 0.$

• From the above eq., we get:

$A = 4q_0L^4/(\pi^5EI)$

• Thus, the approximate solution as per Galerkin method is:

$w(x) = [4q_0L^4/(\pi^5EI)] \cos(\pi x/L)$

• At x = 0, the beam deflection is:

$wGalerkin(0) = 0.01309[q_0L^4/(EI)]$

• Also, the exact solution for beam deflection is:

$wExact(0) = 0.1302\ [q_0L^4/(EI)]$

• Comparing exact and approximate values (as per Galerkin method), we see the two answers are fairly close to each other. To improve the accuracy of the solution we may use more than one term in the assumed solution form. Lets assume,

$w^o(x) = A_1 \cos(\pi x/L) + A_3 \cos(3\pi x/L)$

• Such an assumed expression for w(x) satisfied the kinematic boundary conditions at both ends of the simply supported beam. At this stage, we also assume an expression for virtual displacement.

$w_1(x) = B_1 \cos(\pi x/L) + B_3 \cos(3\pi x/L)$

• The equation for w(x), the equation for virtual displacement, $w_1(x)$, also satisfies kinematic boundary conditions.

• At this stage, we develop an expression for virtual work over the entire domain, and equate it to zero. Thus:

$$\int_{-L/2}^{L/2} E[w(x)]w_1(x)dx = 0$$

or,

$$\int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right]\left(B_1 cos\frac{\pi x}{L} + B_3 cos\frac{3\pi x}{L}\right)dx = 0$$

or,

$$B_1\int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right] cos\frac{\pi x}{L}dx$$

$$+ B_3\int_{-L/2}^{L/2} \epsilon\left[EI\left\{\left(\frac{\pi}{L}\right)^4 A_1 cos\frac{\pi x}{L} + \left(\frac{3\pi}{L}\right)^4 A_3 cos\frac{3\pi x}{L}\right\} - q_o\right] cos\frac{3\pi x}{L}dx = 0$$

But we know that B_1, and B_2 can have arbitrary magnitudes. Hence, integral of virtual work over domain can only be zero, if both terms in [ ] are individually zero.

Thus, we get two parallel equations in A1, and A3. Solving for these equations gives us:

– $A_1 = 4q_0L^4/(\pi 5EI)$

– $A_3 = -A_1/35$

Thus,

$w(x) = [4q_0L_4/(\pi^5 EI)]\ [\cos(\pi x/L) - (1/35)\cos(3\pi x/L)]$.

• At the center, i.e. when x = 0, the value of displacement of the beam is:

$w_{Galerkin}(0) = 0.01309[q_0L^4/(EI)]$ using 1-term solution

$w_{Galerkin}(0) = 0.01302[q_0L^4/(EI)]$ using 2-term solution

$w_{Exact}(0) = 0.1302\ [q_0L^4/(EI)]$ exact solution.

Thus, 2-term solution brings us remarkably close to the exact solution.

Based on this analysis, we make following observations:

– As the number of terms is increased in the assumed form of Galerkin solution, it approaches the exact value monotonically.

– A solution with lesser number of terms represents a stiffer system vis-à-vis a system which has more terms. As number of terms increase, so does the flexibility of the system. Hence, accuracy of solution increases with number of terms.

References

1. Parviz Moin, Engineering Numerical Analysis, Cambridge, University press, 2001.
2. Agmon S, Douglis A, Nirenberg L, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math, 1959, 12; 623-727.
3. Brenner S, Scott LR, The Mathematical Analysis of Finite Element Methods, Springer Verlag, 1994.
4. Roberts JE,Thomas JM, Mixed and Hybrid Methods in Handbook of Numerical Analysis, PG Ciarlet and JL Lions eds, Vol.II, Finite Element Methods (Part 1), North Holland, 1989.
5. Brezzi FG, Manzini D, Marini P, Pietra, Russo A, Discontinuous galerkin approximations for elliptic problems, Numerical Methods for Partial Differential Equations, 2000, 16; 365–378.
6. Chen Z, Nochetto RH, Schmidt A, A characteristic Galerkin method with adaptive error control for the continuous casting problem, Comp. Meth. Appl. Mech. Eng, 2000, 189; 249-276.
7. Glowinski R, Numerical Methods for Nonlinear Variational Problems, Springer, 1984.
8. Kevorkian J, Partial Differential Equations, Chapman and Hall, London, 1990.

9. Menter FR, Two-equation eddy-viscosity turbulence models for engineering applications, AIAA Journal, 1994, 32(8); 1598-1605.
10. Onate E, Idelsohn S, A finite point method in computational mechanics-applications to convective transport, International journal of numerical methods in engineering, 1996, 39(22); 3839-3866.

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