Global Chaos Control of the Generalized Lotka-Volterra Three-Species System via Integral Sliding Mode Control

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Abstract: Since the recent research has shown the importance of biological control in many biological systems appearing in nature, this research paper investigates research in the dynamic and chaotic analysis of the generalized Lotka-Volterra three-species biological system, which was studied by Samardzija and Greller (1988). The generalized Lotka-Volterra biological system consists of two predator and one prey populations. This paper depicts the phase portraits of the 3-D generalized Lotka-Volterra system when the system undergoes chaotic behaviour. Next, this paper derives biological control law via integral sliding mode control (ISMC) for achieving output regulation of the states of the generalized Lotka-Volterra three-species biological system so as to track constant reference signals (set-point controls). All the main results are proved using Lyapunov stability theory. Also, numerical simulations have been shown using MATLAB to illustrate all the main results for the three-species generalized Lotka-Volterra biological system and its output regulation.

Keywords: Chaos, chaotic systems, output regulation, biology, biological system, Lotka-Volterra system, etc.

1. Introduction

Chaos theory describes the qualitative study of deterministic chaotic dynamical systems, and a chaotic system must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [1-2].

The classical chaotic systems are due to Lorenz, who discovered chaos while studying a 3-D weather model in 1963 [3], and Rossler, who discovered chaos, while he was studying chemical reactions in 1976 [4]. These classical systems were followed by the discovery of many 3-D chaotic systems such as Arneodo system [5], Sprott systems [6], Chen system [7], Lü-Chen system [8], Cai system [9], Tigan system [10], etc. Many new chaotic systems have been also discovered in the recent years like Sundarapandian systems [11, 12], Vaidyanathan systems [13-43], Pehlivan system [44], Pham system [45], etc.

In control theory, active control method is used when the parameters are available for measurement [46-65]. Adaptive control is a popular control technique used for stabilizing systems when the system parameters are unknown [66-80]. There are also other popular methods available for control and synchronization of systems such as backstepping control method [81-87], sliding mode control method [88-100], intelligent control [101-110], etc.

Recently, chaos theory is found to have important applications in several areas such as chemistry [111-128], biology [129-160], memristors [161-163], electrical circuits [164], etc.
One of the famous examples of simple biological models is the two-species predator-prey model developed by Lotka and Volterra [165]. Lotka-Volterra system describes the interaction of a two-species predator-prey model and it consists of a system of two nonlinear ordinary differential equations. This is a very popular model and it has many applications of interacting two-species systems. However, this model also has limitations such as it ignores many important factors such as interactions between another species of the same ecosystem, interactions with the environment etc. Thus, three-species models of biological species have more importance. Arneodo et al. [166] have shown that one can obtain chaotic behaviour for three species in an ecosystem. Three species predator-prey models typically consist of one-prey and two predators, and in this research work, we investigate such a three-species biological generalized Lotka-Volterra system investigated by Samardzija and Greller [167].

An agricultural ecosystem comprises a dynamic web of biological relationships among crop plants or trees, herbivores, predators, preys, disease organisms, etc. Organisms in an ecosystem interact in many ways through competition. These organisms constantly evolve and depend on each other and thereby they create a diverse, complex and dynamic environment.

This paper depicts the phase portraits of the 3-D generalized Lotka-Volterra system [167] when the system undergoes chaotic behaviour. Next, this paper derives biological control law via integral sliding mode control (ISMC) for achieving output regulation of the states of the generalized Lotka-Volterra three-species biological system so as to track constant reference signals (set-point controls). All the main results are proved using Lyapunov stability theory. Also, numerical simulations have been plotted using MATLAB to illustrate the main results for the three-species generalized Lotka-Volterra biological system and its output regulation.

2. Generalized Lotka-Volterra Three-Species Biological System

Samardzija and Greller (1988, [167]) derived a generalized Lotka-Volterra three-species biological system, which is described by the 3-D system of differential equations

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2 + cx_1^2 - ax_1^2 x_3 \\
\dot{x}_2 &= -x_2 + x_1 x_2 \\
\dot{x}_3 &= -bx_3 + ax_1^2 x_3
\end{align*}
\]  

In (1), \(x_1\) is the prey population, \(x_2, x_3\) are predator populations and \(a, b, c\) are positive constants. In [145], it was shown that the three-species biological system (1) is chaotic when we take

\[
\begin{align*}
a &= 2.9851, & b &= 3, & c &= 2
\end{align*}
\]  

For numerical simulations, we take the initial conditions as \(x_1(0) = 1.2, x_2(0) = 1.2\) and \(x_3(0) = 1.2\).

The 3-D phase portrait of the generalized Lotka-Volterra system (1) is depicted in Figure 1. The 2-D projections of the generalized Lotka-Volterra systems (2) on the coordinate planes are depicted in Figures 2-3.

![Figure 1. The 3-D phase portrait of the generalized Lotka-Volterra chaotic system](image)
Figure 2. The 2-D projection of the generalized Lotka-Volterra system on \((x_1, x_2)\) plane

Figure 3. The 2-D projection of the generalized Lotka-Volterra system on \((x_2, x_3)\) plane

3. Control of the generalized Lotka-Volterra three-species system via integral sliding mode control

The chaotic behaviour of the generalized Lotka-Volterra three-species biological system [167] is an example of the explosive route to chaos and it is attributed to the non-transversal saddle connection type bifurcation. Also, it is observed that the chaotic solution of the generalized Lotka-Volterra biological system (1) portrays a fractal torus in the 3-D phase space.

In this section, we use integral sliding mode control for controlling the states of the generalized Lotka-Volterra three-species biological system so as to track constant reference signals (set-point controls).

Thus, we consider the generalized Lotka-Volterra system given by the 3-D dynamics

\[
\begin{align*}
\dot{x}_1 &= x_1 - x_1 x_2 + ax_1^2 x_3 + u_1 \\
\dot{x}_2 &= -x_2 + x_1 x_2 + u_2 \\
\dot{x}_3 &= -bx_3 + ax_1^2 x_3 + u_3
\end{align*}
\]

(3)

In (3), \(x_1, x_2, x_3\) are the states, \(a, b, c\) are constant, positive, parameters of the system and \(u_1, u_2, u_3\) are the controls to be determined.

Our design goal is to find controls \(u_1, u_2, u_3\) so as to regulate the states of the system (3) so as to track constant reference signals (set-point controls).

Thus, we define the regulation errors as follows.

\[
\begin{align*}
\epsilon_1(t) &= x_1(t) - \alpha_1 \\
\epsilon_2(t) &= x_2(t) - \alpha_2 \\
\epsilon_3(t) &= x_3(t) - \alpha_3
\end{align*}
\]

(4)

In (4), \(\alpha_1, \alpha_2, \alpha_3\) are constant reference signals or set-point controls.
Then the regulation error dynamics is obtained as
\[
\begin{align*}
\dot{e}_1 &= (e_1 + \alpha_1) - (e_1 + \alpha_1)(e_1 + \alpha_1) + c(e_1 + \alpha_1)^2 - a(e_1 + \alpha_1)^2(e_2 + \alpha_2) + u_1 \\
\dot{e}_2 &= -(e_2 + \alpha_2) + (e_1 + \alpha_1)(e_2 + \alpha_2) + u_2 \\
\dot{e}_3 &= -b(e_3 + \alpha_3) + a(e_1 + \alpha_1)^2(e_3 + \alpha_3) + u_3
\end{align*}
\]
(5)

Based on the sliding mode control theory [168], the integral sliding surface of each error variable \( e_i \), \( i = 1, 2, 3 \) is defined as follows:
\[
s_i = \left( \frac{d}{dt} + \lambda_i \right) \left[ \int_0^{t} e_i(\tau) d\tau \right] = e_i + \lambda_i \int_0^{t} e_i(\tau) d\tau, \quad (i = 1, 2, 3)
\]
(6)

The derivative of each equation in (6) yields
\[
\dot{s}_i = \dot{e}_i + \lambda_i e_i, \quad (i = 1, 2, 3)
\]
(7)

The Hurwitz condition is satisfied if \( \lambda_i > 0 \) for \( i = 1, 2, 3 \).

Based on the exponential reaching law [146], we set
\[
s_i = -\eta_i \text{sgn}(s_i) - k_i s_i, \quad (i = 1, 2, 3)
\]
(8)

where \( \eta_i, k_i, (i = 1, 2, 3) \) are positive constants.

Comparing the equations (7) and (8), we get
\[
\begin{align*}
\dot{e}_1 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i \\
\dot{e}_2 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i \\
\dot{e}_3 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i
\end{align*}
\]
(9)

Using Eq. (5), we can rewrite Eq. (9) as follows.
\[
\begin{align*}
x_1 - x_1 x_2 + cx_2^2 - ax_1^2 x_3 + u_1 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i \\
x_2 - x_1 x_2 + u_2 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i \\
x_3 + ax_1^2 x_3 + u_3 + \lambda_i e_i &= -\eta_i \text{sgn}(s_i) - k_i s_i
\end{align*}
\]
(10)

From Eq. (10), the control laws are obtained as follows:
\[
\begin{align*}
u_1 &= -x_1 + x_1 x_2 - cx_2^2 + ax_1^2 x_3 - \lambda_i e_1 - \eta_i \text{sgn}(s_i) - k_i s_i \\
u_2 &= x_2 - x_1 x_2 - \lambda_i e_2 - \eta_i \text{sgn}(s_i) - k_i s_i \\
u_3 &= bx_3 - ax_1^2 x_3 - \lambda_i e_3 - \eta_i \text{sgn}(s_i) - k_i s_i
\end{align*}
\]
(11)

Next, we state and prove the main result of this section.

**Theorem 1.** The states \( x_1, x_2, x_3 \) of the generalized Lotka-Volterra three-species system (3) are regulated to track the constant reference signals \( \alpha_1, \alpha_2, \alpha_3 \) asymptotically as \( t \to \infty \) for all initial conditions \( x(0) \in \mathbb{R}^3 \) by the integral sliding mode control law (11), where the constants \( \lambda_i, \eta_i, k_i \) are positive for \( i = 1, 2, 3 \).

**Proof.** This result is proved using Lyapunov stability theory [169].

We consider the following quadratic Lyapunov function
\[
V(s_1, s_2, s_3) = \frac{1}{2} \left( s_1^2 + s_2^2 + s_3^2 \right)
\]
(12)

where \( s_1, s_2, s_3 \) are as defined in Eq. (6). The time-derivative of \( V \) is obtained as
\[
\dot{V} = s_1 \dot{s}_1 + s_2 \dot{s}_2 + s_3 \dot{s}_3
\]
(13)

Substituting from Eq. (8) into Eq. (13), we obtain
\[
\dot{V} = s_1 (-\eta_1 \text{sgn}(s_1) - k_1 s_1) + s_2 (-\eta_2 \text{sgn}(s_2) - k_2 s_2) + s_3 (-\eta_3 \text{sgn}(s_3) - k_3 s_3)
\]
(14)

Simplifying Eq. (14), we obtain
\[
\dot{V} = -\eta_1 |s_1| - k_1 s_1^2 - \eta_2 |s_2| - k_2 s_2^2 - \eta_3 |s_3| - k_3 s_3^2
\]
(15)
Since $k_i > 0$ and $\eta_i > 0$ for $i = 1, 2, 3$, it follows from (15) that $\dot{V}$ is a negative definite function.

Thus, by Lyapunov’s stability theory [169], it is immediate that the error $e_i(t) \to 0$, $(i = 1, 2, 3)$ asymptotically as $t \to \infty$ for all initial conditions $e(0) \in \mathbb{R}^3$.

This completes the proof. ■

4. Numerical Simulations

We use classical fourth-order Runge-Kutta method in MATLAB with step-size $h = 10^{-8}$ for solving the system of differential equations (3) when the integral sliding mode controller (11) is implemented.

The parameter values of the generalized Lotka-Volterra system (3) are taken as in the chaotic case, viz.

\[
a = 2.9851, \quad b = 3, \quad c = 2
\]

We take the sliding constants as

\[
\eta_i = \lambda_i = 0.1, \quad k_i = 30, \quad (i = 1, 2, 3)
\]

We take the initial conditions of the generalized Lotka-Volterra system (3) as

\[
x_i(0) = 12.9, \quad x_2(0) = 14.1, \quad x_3(0) = 20.8
\]

We take the constant reference signals as

\[
\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 3
\]

Figures 4-6 show the output regulation of the generalized Lotka-Volterra system (3).

Figure 7 shows the time-history of the regulation errors $e_1, e_2, e_3$.

![Figure 4. Output regulation of the state $x_i$](image)
5. Conclusions

In this paper, new results have been derived for the analysis and adaptive synchronization of the three-
species generalized Lokta-Volterra biological systems discovered by Samardzića and Greller (1988). After a description and dynamic analysis of the chaotic 3-D three species Samardzića-Greller model, we have designed a biological feedback controller for the global and asymptotic state regulation of the states of the 3-D three species Samardzića-Greller model to constant reference signals (set-point controls) via integral sliding mode control. The main results have been proved using Lyapunov stability theory and numerical simulations have been illustrated using MATLAB.

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