Coleman-Gomatam Logarithmic Competitive Biology Models and Their Ecological Monitoring

Sundarapandian Vaidyanathan

R & D Centre, Vel Tech University, Avadi, Chennai, Tamil Nadu, INDIA

Abstract: Lotka-Volterra population biology models are important models that describe the interaction between various biological species. Some important Lotka-Volterra population biology models are predator-prey models and competitive biology models. There are certain classes of biological models like the drosophila population models for which the dynamic behaviour cannot be sufficiently described by the Lotka-Volterra population biology models. In this research work, we study the Coleman-Gomatam logarithmic competitive biology models (1972). We show that for the Coleman-Gomatam logarithmic biological model, under an assumption, the two competing species have stable coexistence. Then we shall propose ecological monitoring of the Coleman-Gomatam logarithmic competitive population biology models by constructing nonlinear observers for them about their stable equilibrium points. The nonlinear observer design for the population biology model is constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the Coleman-Gomatam logarithmic competitive population biology model and the population size of any of the competitive species as the output function. Numerical examples are given to illustrate the ecological monitoring or the nonlinear observer design for the Coleman-Gomatam two-species logarithmic competitive biology model with stable coexistence. MATLAB simulations are shown to illustrate the numerical results shown in this research work.

Keywords: Population biology, Lotka-Volterra model, Coleman-Gomatam model, logarithmic competitive model, two-species model, ecological monitoring, observer design, etc.

1. Introduction

Lotka-Volterra population biology models are important models that describe the interaction between various biological species considered as predator-prey system [1-2]. In the population biology literature, Lotka-Volterra two species competitive biology model is also well-known for the study of competing population models [3]. There are certain classes of biological models like the drosophila population models for which the dynamic behaviour cannot be sufficiently described by the Lotka-Volterra population biology models. In this work, we study the Coleman-Gomatam logarithmic competitive biology models ([4], 1972). We show that for this biological model, under an assumption, the two competing species have stable co-existence. After discussion on the Coleman-Gomatam logarithmic competitive biology models, we propose ecological monitoring of the Coleman-Gomatam logarithmic competitive biology models by explicitly constructing local exponential observers for the Coleman-Gomatam logarithmic competitive biology models.

In control systems engineering, a *state observer* is a system that provides an estimate of the internal state of a given real system, from measurements of the input and output of the real system. It is typically

computer-implemented, and provides the basis of many practical applications.

The problem of designing observers for linear control systems was first proposed and fully solved by Luenberger [5]. The problem of designing observers for nonlinear control systems was proposed by Thau [6]. Over the past three decades, significant attention has been paid in the control systems literature on the construction of observers for nonlinear control systems [7].

A characterization of local exponential observers for nonlinear control systems was first obtained by Sundarapandian [8]. In [8], necessary and sufficient conditions were obtained for exponential observers for Lyapunov stable continuous-time nonlinear systems. In [8], an exponential observer design was provided by Sundarapandian for nonlinear control systems, which generalizes the linear observer design of Luenberger [4] for linear control systems.

In [9], Sundarapandian obtained necessary and sufficient conditions for exponential observers for Lyapunov stable discrete-time nonlinear systems and also provided a formula for designing exponential observers for Lyapunov stable discrete-time nonlinear systems. In [10], Sundarapandian derived new results for the global observer design for nonlinear control systems.

The concept of nonlinear observers for nonlinear control systems was extended in many ways. In [11-12], Sundarapandian derived new results for the characterization of local exponential observers for nonlinear bifurcating systems. In [13-16], Sundarapandian derived new results for the exponential observer design for a general class of nonlinear systems with real parametric uncertainty. In [17-20], Sundarapandian derived new results for the general observers for nonlinear systems. In [21], Sundarapandian derived new results for observers around equilibria. In [22-23], Sundarapandian derived new results for periodic orbits of nonlinear control systems.

In this work, we discuss the properties and ecological monitoring of the two-species Coleman-Gomatam logarithmic competitive biology models. In Section 2, we review the definition and results of local exponential observers for nonlinear systems. In Section 3, we describe the two species Coleman-Gomatam logarithmic competitive biology models. In Section 4, we describe the ecological monitoring or the exponential observer design of the two-species Coleman-Gomatam logarithmic competitive biology model. Section 5 contains the conclusions of this research work.

2.   Review of Nonlinear Observer Design for Nonlinear Systems

An *observer* for a nonlinear system is a state estimator, and the states of the observer converge to the states of the plant dynamics asymptotically or exponentially as time tends to infinity.

We consider the nonlinear system described by

$$\dot{x} = f(x) \tag{1a}$$
$$y = h(x) \tag{1b}$$

where $x \in R^n$ is the *state* and $y \in R^p$ is the *output*.

We assume that $f : R^n \to R^n$, $h : R^n \to R^p$ are C^1 mappings and for some $x^* \in R^n$, the following hold:

$$f(x^*) = 0, \quad h(x^*) = 0 \tag{2}$$

Remark 1. The solutions x^* of $f(x) = 0$ are called the *equilibrium points* of the system dynamics (1a). Also, the assumption $h(x^*) = 0$ holds without any loss of generality. Indeed, if $h(x^*) \neq 0$, then we can define a new output function as

$$\psi(x) = h(x) - h(x^*) \tag{3}$$

and it is easy to see that $\psi(x^*) = 0$.   ■

The linearization of the nonlinear system (1a)-(1b) at $x = x^*$ is given by

$$\dot{x} = Ax \tag{4a}$$
$$y = Cx \tag{4b}$$

where

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=x^*} \quad \text{and} \quad C = \left[\frac{\partial h}{\partial x}\right]_{x=x^*} \tag{5}$$

Definition 1. [21] A C^1 dynamical system defined by

$$\dot{z} = g(z, y), \quad (z \in R^n) \tag{6}$$

is called a **local asymptotic** (respectively, **local exponential**) observer for the nonlinear system (1a)-(1b) if the following two requirements are satisfied:

(O1)   If $z(0) = x(0)$, then $z(t) = x(t)$, for all $t \geq 0$.

(O2)   There exists a neighbourhood V of the equilibrium $x^* \in R^n$ such that for all $z(0), x(0) \in V$, the estimation error

$$e(t) = z(t) - x(t) \tag{7}$$

decays asymptotically (respectively, exponentially) to zero as $t \to \infty$. ∎

Theorem 1. (Sundarapandian, [21]) Suppose that the nonlinear system dynamics (1a) is Lyapunov stable at the equilibrium $x = x^*$ and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the dynamical system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{8}$$

is a local exponential observer for the nonlinear system (1a)-(1b).   ∎

Remark 2. The estimation error is governed by the error dynamics

$$\dot{e} = f(x+e) - f(x) - K[h(x+e) - h(x)] \tag{9}$$

Linearizing the error dynamics (9) at $x = x^*$, we get the linear system

$$\dot{e} = Ee, \qquad \text{where} \quad E = A - KC \tag{10}$$

If (C, A) is observable, then the eigenvalues of the error matrix $E = A - KC$ can be arbitrarily placed in the complex plane. Thus, when (C, A) is observable, a local exponential observer of the form (8) can be always found such that the transient response of the error decays quickly with any desired speed of convergence. ∎

3.   Coleman-Gomatam Two Species Logarithmic Competitive Biology Models

In this section, we consider the Coleman-Gomatam two-species logarithmic competitive biology system [4], which is modeled by the system of differential equations

$$\begin{cases} \dot{x}_1 = x_1(a_1 - b_{11} \ln x_1 - b_{12} \ln x_2) \\ \dot{x}_2 = x_2(a_2 - b_{21} \ln x_1 - b_{22} \ln x_2) \end{cases} \tag{11}$$

In (11), $a_1, a_2, b_{ij}(i, j = 1, 2)$ are positive constants, and x_1, x_2 are population densities, which are greater than one. We note that a_1, a_2 are the growth rate constants for the species 1 and 2 respectively. One of the reasons for using the logarithmic terms $x_i \ln x_j$ in (11) instead of the quadratic terms $x_i x_j$ as in Lotka-Volterra model is that the presence of a large x_j suppresses the effectiveness of x_i and so, a slowly increasing $\ln x_j$ is preferable in a two-species competitive biology model.

In this paper, we suppose that the competing species (11) satisfies the following assumption:

(H) $\dfrac{b_{12}}{b_{22}} < \dfrac{b_{11}}{b_{21}}$

The equilibrium points of the system (11) are obtained by solving the system of equations

$$\begin{cases} x_1\left(a_1 - b_{11}\ln x_1 - b_{12}\ln x_2\right) = 0 \\ x_2\left(a_2 - b_{21}\ln x_1 - b_{22}\ln x_2\right) = 0 \end{cases} \tag{12}$$

Clearly, $E_1(0,0), E_2(0, e^{a_2/b_{22}}), E_3(e^{a_1/b_{11}}, 0)$ are three equilibrium points of the system (11).

We are interested in finding a positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ of the system (11), which is obtained by solving the system of equations (12) with the conditions that $x_1^* > 0$ and $x_2^* > 0$. Thus, we obtain the positive equilibrium point $E_4\left(x_1^*, x_2^*\right)$ by solving the equations

$$\begin{cases} a_1 - b_{11}\ln x_1 - b_{12}\ln x_2 = 0 \\ a_2 - b_{21}\ln x_1 - b_{22}\ln x_2 = 0 \end{cases} \tag{13}$$

Define

$$y_1 = \ln x_1 \text{ and } y_2 = \ln x_2. \tag{14}$$

Then the system (13) can be expressed as the linear system

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{15}$$

The system (15) has a unique solution if and only if the coefficient matrix is non-singular, i.e.

$$\Delta \quad b_{11}b_{22} - b_{12}b_{21} \neq 0 \tag{16}$$

By assumption (H), we know that

$$\frac{b_{12}}{b_{22}} < \frac{b_{11}}{b_{21}} \quad \text{or} \quad b_{11}b_{22} - b_{12}b_{21} > 0 \tag{17}$$

Thus, we can use Cramer's rule to solve the linear system of equations (15). We find that

$$\Delta_1 \quad \begin{vmatrix} a_1 & b_{12} \\ a_2 & b_{22} \end{vmatrix} = a_1 b_{22} - a_2 b_{12} \text{ and } \Delta_2 \quad \begin{vmatrix} b_{11} & a_1 \\ b_{21} & a_2 \end{vmatrix} = a_2 b_{11} - a_1 b_{21} \tag{18}$$

Since $\Delta \neq 0$, we get the unique solution of the linear system (15) as

$$y_1 = \frac{\Delta_1}{\Delta} = \frac{a_1 b_{22} - a_2 b_{12}}{b_{11}b_{22} - b_{12}b_{21}} \quad \text{and} \quad y_2 = \frac{\Delta_2}{\Delta} = \frac{a_2 b_{11} - a_1 b_{21}}{b_{11}b_{22} - b_{12}b_{21}} \tag{19}$$

Thus, the positive equilibrium of the system (11) is obtained as

$$x_1^* = e^{y_1^*} \quad \text{and} \quad x_2^* = e^{y_2^*} \tag{20}$$

The Jacobian or community matrix corresponding to $E_4(x_1^*, x_2^*)$ is obtained as

$$A = \begin{pmatrix} -b_{11} & -\dfrac{b_{12}x_1^*}{x_2^*} \\ -\dfrac{b_{21}x_2^*}{x_1^*} & -b_{22} \end{pmatrix} \tag{21}$$

Next, we find the characteristic equation of the community matrix A as

$$\lambda^2 - \text{Trace}(A)\lambda + \det(A) = 0 \tag{22}$$

We note that

$$\text{Trace}(A) = -b_{11} - b_{22} < 0 \tag{23}$$

$$\det(A) = b_{11}b_{22} - b_{12}b_{21} > 0 \tag{24}$$

Since all the coefficients of the quadratic equation (22) are positive, it is immediate from Hurwitz criterion [22] that all the eigenvalues of the community matrix A are stable.

Thus, A is a Hurwitz matrix.

Thus, from Lyapunov stability theory [24], it is immediate that the positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ is locally asymptotically stable. Hence, we have proved the following theorem.

Theorem 2. Suppose that the Coleman-Gomatam two-species logarithmic competitive biology system (11) satisfies the assumption (H) stated as follows.

(H) $\dfrac{b_{12}}{b_{22}} < \dfrac{b_{11}}{b_{21}}$

Then $E_4\left(x_1^*, x_2^*\right)$ is a positive equilibrium of the system (11). Also, the unique positive equilibrium $E_4\left(x_1^*, x_2^*\right)$ of the Lotka-Volterra population biology system (11) is locally asymptotically stable. ∎

4. Ecological Monitoring for the Coleman-Gomatam Two-Species Competitive Biology Systems

In this section, we discuss how to do ecological monitoring of the Coleman-Gomatam two-species logarithmic competitive biology systems by designing a local exponential observer to estimate their states.

4.1 Ecological Monitoring of the Competitive Models with Population Density of Species 1 as Output

We consider the Coleman-Gomatam two-species logarithmic competition biology system given by

$$\begin{cases} \dot{x}_1 = x_1(a_1 - b_{11}\ln x_1 - b_{12}\ln x_2) \\ \dot{x}_2 = x_2(a_2 - b_{21}\ln x_1 - b_{22}\ln x_2) \end{cases} \tag{25}$$

We suppose that the population density of species 1 is given as the system output, *i.e.*

$$y = x_1 \tag{26}$$

We suppose that the assumption (H) holds so that $\left(x_1^*, x_2^*\right)$ is a unique positive equilibrium of the system (25).

In Section 3, we showed that the community matrix of the system (25) about the unique positive equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$A = \begin{pmatrix} -b_{11} & -\dfrac{b_{12}x_1^*}{x_2^*} \\ -\dfrac{b_{21}x_2^*}{x_1^*} & -b_{22} \end{pmatrix}, \tag{27}$$

which is a Hurwitz matrix. Thus, the equilibrium $\left(x_1^*, x_2^*\right)$ is locally asymptotically stable.

Moreover, the linearization of the output function (26) about the equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{28}$$

Thus, the observability matrix for the system (25)-(26) is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -b_{11} & -\dfrac{b_{12}x_1^*}{x_2^*} \end{bmatrix} \tag{29}$$

We find that

$$`\det(W) = -\dfrac{b_{12}x_1^*}{x_2^*} \neq 0 \tag{30}$$

which shows that the observability matrix W has full rank.
Thus, by Kalman's rank test for observability [25], the system (25)-(26) is completely observable.

Hence, by Sundarapandian's theorem (Theorem 1, Section 2), we obtain the following main result, which gives the ecological monitoring of the Coleman-Gomatam two-species logarithmic competition biology systems.

Theorem 3. Suppose that the assumption (H) is satisfied. Then the Coleman-Gomatam two-species logarithmic competitive biology system (25) with output (26) has a local exponential observer of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1\left(a_1 - b_{11}\ln z_1 - b_{12}\ln z_2\right) \\ z_2\left(a_2 - b_{21}\ln z_1 - b_{22}\ln z_2\right) \end{bmatrix} + K\left[y - z_1\right] \tag{31}$$

where K is a matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, an observer gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of stable eigenvalues. ∎

Example 1. We consider a two species Coleman-Gomatam logarithmic competitive biology system given by

$$\begin{cases} \dot{x}_1 = x_1(31 - 8\ln x_1 - 5\ln x_2) \\ \dot{x}_2 = x_2(42 - 9\ln x_1 - 8\ln x_2) \end{cases} \tag{32}$$

where $x_1 \geq 1$ and $x_2 \geq 1$.

Suppose that the output function given by the density of competing species 1, *i.e.*

$$y = x_1 \tag{33}$$

Here,

$$a_1 = 31, \ a_2 = 42, \ b_{11} = 8, \ b_{12} = 5, \ b_{21} = 9, \ b_{22} = 8 \tag{34}$$

We note that

$$\frac{b_{12}}{b_{22}} = \frac{5}{8} < \frac{b_{11}}{b_{21}} = \frac{8}{9} \tag{35}$$

Thus, the assumption (H) is satisfied.

We find the positive equilibrium of the system (32) by solving the equations

$$\begin{cases} x_1(31 - 8\ln x_1 - 5\ln x_2) = 0 \\ x_2(42 - 9\ln x_1 - 8\ln x_2) = 0 \end{cases} \tag{36}$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we obtain

$$\begin{cases} 31 - 8\ln x_1 - 5\ln x_2 = 0 \\ 42 - 9\ln x_1 - 8\ln x_2 = 0 \end{cases} \tag{37}$$

We define

$$y_1 = \ln x_1, \ y_2 = \ln x_2 \tag{38}$$

Then the system (37) can be easily arranged in matrix form as

$$\begin{bmatrix} 8 & 5 \\ 9 & 8 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 31 \\ 42 \end{bmatrix} \tag{39}$$

By solving the linear system (39), we get the unique solution as

$$y_1^* = 2, \ y_2^* = 3 \tag{40}$$

Then the positive equilibrium $\left(x_1^*, x_2^*\right)$ of the system (32) is obtained as

$$x_1^* = e^{y_1^*} = e^2 = 7.3891, \qquad x_2^* = e^{y_2^*} = e^3 = 20.0855 \tag{41}$$

As shown in Section 3, the Coleman-Gomatam logarithmic competitive population biology system (32) is locally asymptotically stable about the unique positive equilibrium point $\left(x_1^*, x_2^*\right)$.

The linearization of the Coleman-Gomatam population biology dynamics (32) at $\left(x_1^*, x_2^*\right)$ is given by

$$A = \begin{bmatrix} -8 & -1.8394 \\ -24.4645 & -8 \end{bmatrix} \tag{42}$$

Also, the linearization of the output function (33) at $\left(x_1^*, x_2^*\right)$ is given by

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{43}$$

It is easy to check that the observability matrix $W = \begin{bmatrix} C \\ CA \end{bmatrix}$ has full rank.

This shows that the given system (32) with output (33) is completely observable near the positive equilibrium

point $\left(x_1^*, x_2^*\right) = (7.3891, 20.0855)$.

For numerical simulations, we take $x_1(0) = 5.2$ and $x_2(0) = 2.8$.

Figure 1 illustrates that the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (7.3891,\ 20.0855)$ is locally asymptotically stable.
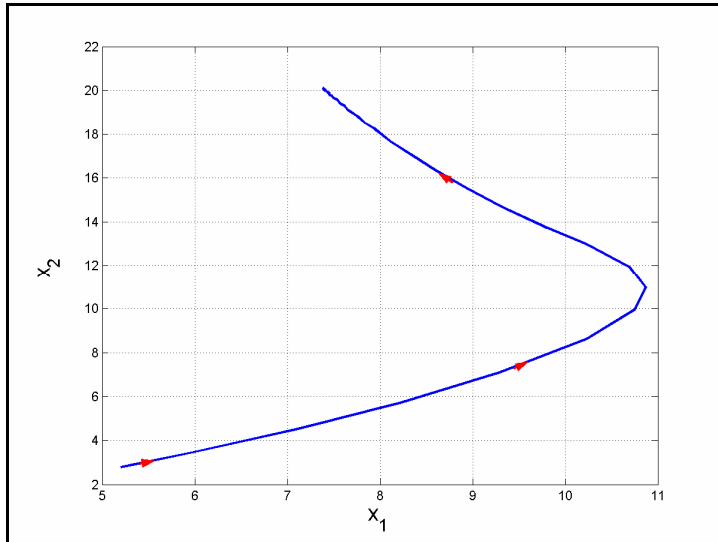

Figure 1. State Orbit of the Coleman-Gomatam Competition Biology System (32)

Since (C, A) is observable, the eigenvalues of the error matrix $E = A - KC$ can be placed arbitrarily.

Using the Ackermann's formula [25] for the observer gain matrix, we can choose K so that the error matrix $E = A - KC$ has the stable eigenvalues $\{-6, -6\}$.

A simple calculation using MATLAB gives

$$K = \begin{bmatrix} -4.0000 \\ -26.6391 \end{bmatrix} \tag{44}$$

By Theorem 3, a local exponential observer for the Coleman-Gomatam logarithmic competitive biology system (32)-(33) around the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (7.3891,\ 20.0855)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(31 - 8\ln z_1 - 5\ln z_2) \\ z_2(42 - 9\ln z_1 - 8\ln z_2) \end{bmatrix} + \begin{bmatrix} -4.0000 \\ -26.6391 \end{bmatrix} [y - z_1] \tag{45}$$

For simulations, we choose the initial conditions of the plant dynamics (32) as

$$x_1(0) = 2, \quad x_2(0) = 3 \tag{46}$$

Also, we choose the initial conditions of the observer dynamics (45) as

$$z_1(0) = 10, \quad z_2(0) = 4 \tag{47}$$

Figures 2-3 depict the exponential convergence of the observer states z_1 and z_2 of the system (45) to the states x_1 and x_2 of the Coleman-Gomatam logarithmic competitive biology system (32)-(33).
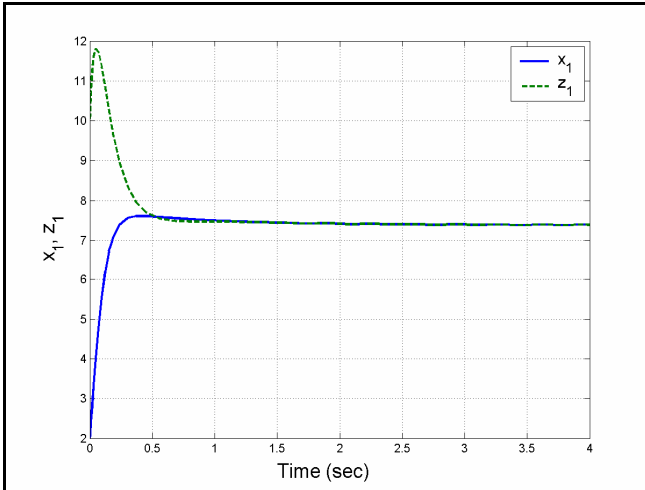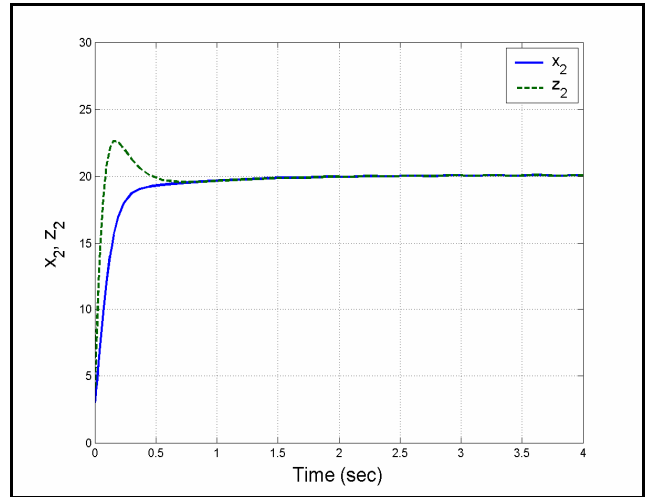
Figure 2. Synchronization of the states x_1 and z_1

Figure 3. Synchronization of the states x_2 and z_2

4.1 Ecological Monitoring of the Competitive Models with Population Density of Species 2 as Output

We consider the Coleman-Gomatam two-species logarithmic competition biology system given by

$$\begin{cases} \dot{x}_1 = x_1(a_1 - b_{11} \ln x_1 - b_{12} \ln x_2) \\ \dot{x}_2 = x_2(a_2 - b_{21} \ln x_1 - b_{22} \ln x_2) \end{cases} \tag{48}$$

We suppose that the population density of species 2 is given as the system output, *i.e.*

$$y = x_2 \tag{49}$$

We suppose that the assumption (H) holds so that $\left(x_1^*, x_2^*\right)$ is a unique positive equilibrium of the system (48).

In Section 3, we showed that the community matrix of the system (48) about the unique positive equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$A = \begin{pmatrix} -b_{11} & -\dfrac{b_{12}x_1^*}{x_2^*} \\ -\dfrac{b_{21}x_2^*}{x_1^*} & -b_{22} \end{pmatrix}, \tag{50}$$

which is a Hurwitz matrix. Thus, the equilibrium $\left(x_1^*, x_2^*\right)$ is locally asymptotically stable.

Moreover, the linearization of the output function (49) about the equilibrium $\left(x_1^*, x_2^*\right)$ is given by

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{51}$$

Thus, the observability matrix for the system (48)-(49) is given by

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{b_{21}x_2^*}{x_1^*} & -b_{22} \end{bmatrix} \tag{52}$$

We find that

$$\text{`det}(W) = \frac{b_{21}x_2^*}{x_1^*} \neq 0 \tag{53}$$

which shows that the observability matrix W has full rank.

Thus, by Kalman's rank test for observability [25], the system (48)-(49) is completely observable.

Hence, by Sundarapandian's theorem (Theorem 1, Section 2), we obtain the following main result, which gives the ecological monitoring of the Coleman-Gomatam two-species logarithmic competition biology systems.

Theorem 4. Suppose that the assumption (H) is satisfied. Then the Coleman-Gomatam two-species logarithmic competitive biology system (48) with output (49) has a local exponential observer of the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1 \left(a_1 - b_{11} \ln z_1 - b_{12} \ln z_2 \right) \\ z_2 \left(a_2 - b_{21} \ln z_1 - b_{22} \ln z_2 \right) \end{bmatrix} + K \begin{bmatrix} y - z_2 \end{bmatrix} \tag{54}$$

where K is a matrix chosen such that $A - KC$ is Hurwitz. Since (C, A) is observable, an observer gain matrix K can be found such that the error matrix $E = A - KC$ has arbitrarily assigned set of stable eigenvalues. ∎

Example 2. We consider a two species Coleman-Gomatam logarithmic competitive biology system given by

$$\begin{cases} \dot{x}_1 = x_1 (34 - 6 \ln x_1 - 5 \ln x_2) \\ \dot{x}_2 = x_2 (52 - 8 \ln x_1 - 10 \ln x_2) \end{cases} \tag{55}$$

where $x_1 \geq 1$ and $x_2 \geq 1$.

Suppose that the output function given by the density of competing species 2, *i.e.*

$$y = x_2 \tag{56}$$

Here,

$$a_1 = 34, \ a_2 = 52, \ b_{11} = 6, \ b_{12} = 5, \ b_{21} = 8, \ b_{22} = 10 \tag{57}$$

We note that

$$\frac{b_{12}}{b_{22}} = \frac{1}{2} < \frac{b_{11}}{b_{21}} = \frac{3}{4} \tag{58}$$

Thus, the assumption (H) is satisfied.
We find the positive equilibrium of the system (55) by solving the equations

$$\begin{cases} x_1 (34 - 6 \ln x_1 - 5 \ln x_2) = 0 \\ x_2 (52 - 8 \ln x_1 - 10 \ln x_2) = 0 \end{cases} \tag{59}$$

Since $x_1 \neq 0$ and $x_2 \neq 0$, we obtain

$$\begin{cases} 34 - 6 \ln x_1 - 5 \ln x_2 = 0 \\ 52 - 8 \ln x_1 - 10 \ln x_2 = 0 \end{cases} \tag{60}$$

We define

$$y_1 = \ln x_1, \ y_2 = \ln x_2 \tag{61}$$

Then the system (60) can be easily arranged in matrix form as

$$\begin{bmatrix} 6 & 5 \\ 8 & 10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 34 \\ 52 \end{bmatrix} \tag{62}$$

By solving the linear system (62), we get the unique solution as

$$y_1^* = 4, \ y_2^* = 2 \tag{63}$$

Then the positive equilibrium $\left(x_1^*, x_2^* \right)$ of the system (55) is obtained as

$$x_1^* = e^{y_1^*} = e^4 = 54.5982, \qquad x_2^* = e^{y_2^*} = e^2 = 7.3891 \tag{64}$$

As shown in Section 3, the Coleman-Gomatam logarithmic competitive population biology system (55) is locally asymptotically stable about the unique positive equilibrium point $\left(x_1^*, x_2^* \right)$.

The linearization of the Coleman-Gomatam population biology dynamics (55) at $\left(x_1^*, x_2^* \right)$ is given by

$$A = \begin{bmatrix} -6 & -36.9453 \\ -1.0827 & -10 \end{bmatrix} \tag{65}$$

Also, the linearization of the output function (56) at $\left(x_1^*, x_2^* \right)$ is given by

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{66}$$

It is easy to check that the observability matrix $W = \begin{bmatrix} C \\ CA \end{bmatrix}$ has full rank. This shows that the given system (55)

with output (56) is completely observable near the positive equilibrium point $\left(x_1^*, x_2^*\right) = (54.5982, 7.3891)$.
For numerical simulations, we take $x_1(0) = 11.7$ and $x_2(0) = 1.5$.

Figure 4 illustrates that the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (54.5982, 7.3891)$ is locally asymptotically stable.
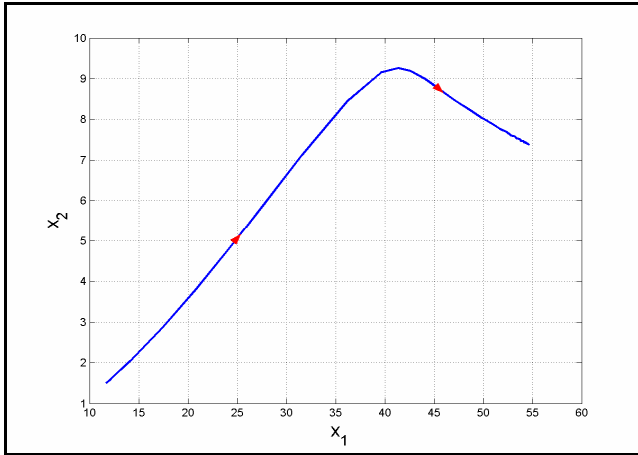

Figure 4. State Orbit of the Coleman-Gomatam Competition Biology System (55)

Since (C, A) is observable, the eigenvalues of the error matrix $E = A - KC$ can be placed arbitrarily.
Using the Ackermann's formula [25] for the observer gain matrix, we can choose K so that the error matrix
$E = A - KC$ has the stable eigenvalues $\{-6, -6\}$.
A simple calculation using MATLAB gives

$$K = \begin{bmatrix} -36.9453 \\ -4.0000 \end{bmatrix}. \tag{67}$$

By Theorem 4, a local exponential observer for the Coleman-Gomatam logarithmic competitive biology system
(55)-(56) around the unique positive equilibrium point $\left(x_1^*, x_2^*\right) = (54.5982, 7.3891)$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(34 - 6\ln z_1 - 5\ln z_2) \\ z_2(52 - 8\ln z_1 - 10\ln z_2) \end{bmatrix} + \begin{bmatrix} -36.9453 \\ -4.0000 \end{bmatrix} [y - z_2] \tag{68}$$

For simulations, we choose the initial conditions of the plant dynamics (55) as
$$x_1(0) = 12, \quad x_2(0) = 5 \tag{69}$$
Also, we choose the initial conditions of the observer dynamics (68) as
$$z_1(0) = 7, \quad z_2(0) = 9 \tag{70}$$

Figures 5-6 depict the exponential convergence of the observer states z_1 and z_2 of the system (68) to the
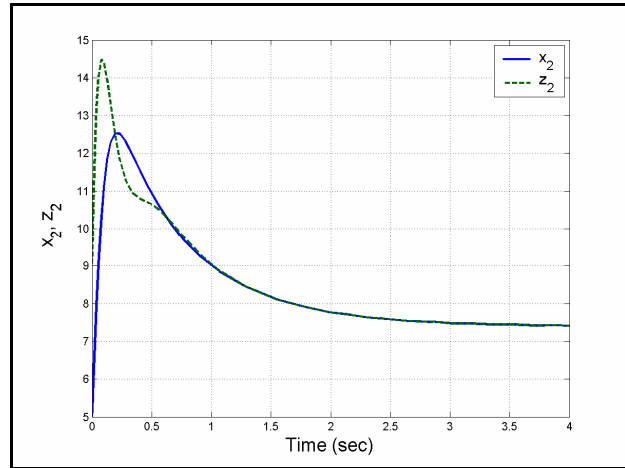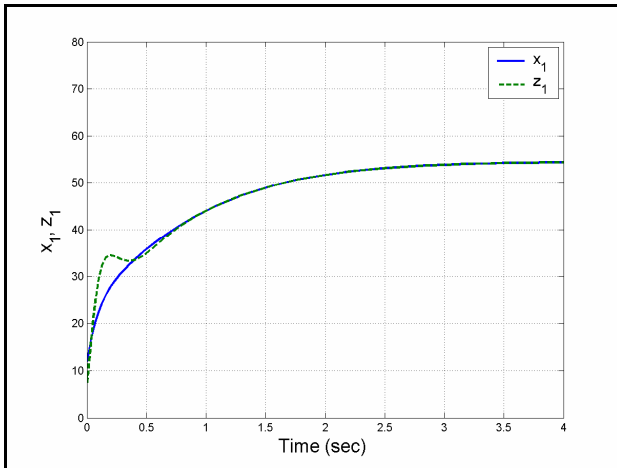states x_1 and x_2 of the Coleman-Gomatam logarithmic competitive biology system (55)-(56).

Figure 5. Synchronization of the states x_1 **and** z_1 **Figure 6. Synchronization of the states** x_2 **and** z_2

5.   Conclusions

In this paper, we described Coleman-Gomatam two-species logarithmic competitive biology models. We showed that for this biological model, under an assumption, the two competing species have stable coexistence. Then we achieved ecological monitoring of the population biology model by constructing a nonlinear exponential observer for the competitive biology model under study. The nonlinear observer design for the competitive biology model was constructed by applying Sundarapandian's theorem (2002) and using only the dynamics of the Coleman-Gomatam two-species logarithmic competitive population biology model and any of the density of the two competing species as the output function. Numerical example and MATLAB simulations were shown to illustrate the ecological monitoring or the nonlinear observer design for the two-species Coleman-Gomatam logarithmic competitive biology models.

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